The Difficulties Semi-Linear in Low dimensions between Brezis and Nirenberg

Dr. BUSHARA EISA HAMAD ABDALLA
Assistant of professor Mathematics, Department of Mathematics
White Nile University, Kosti, Sudan

الملخص

تناقش هذة الدراسة نتائج للمعادلة الدائرية شبة الخطية لنمو سوبوليف الحرج$[3,14]$ في الحالة الحرجة منخفضة الابعاد عندما تكون ذات خاصية شاملة والتي يرمز لها بخصائص رمزية وذلك لتطابق قانون دالة قرين كما في .$[4,9]$

Abstract:

This study discusses existence results for a quasi-linear elliptic equation of critical Sobolev growth [3,14] in the low-dimensional case, where the problem has a global character which is encoded in sign properties of the "regular" part for the corresponding Green's function as in [9,11].

Keywords: Brezis- Nirenberg problem, Low dimensions, (1+ϵ)-Laplace operator, Blow-up.

E-mail address: Boshara@wnu.edu.sd

1.Introduction.

Let Ψ be a bounded in $\mathbb{R}^N$, $N > 2$. Given $0 < \epsilon < N - 1$ and $\ltimes < \ltimes_1$, let us discuss existence issues for the quasilinear problem

$$\begin{cases} -\Delta_{1+\epsilon}\, u = \ltimes_{u^{\epsilon}} + u^{(1+\epsilon)^*-1} & \text{in } \Psi \\ u > 0 & \text{in } \Psi \\ u = 0 & \text{on } \partial\Psi, \end{cases} \qquad (1)$$

where $\Delta_{1+\epsilon}(\cdot) = (\mathrm{div}\, |\nabla(\cdot)|^{\epsilon-1}\nabla(\cdot))$ is the $(1+\epsilon)-$ Laplace operator, $(1+\epsilon)^* = \frac{N(1+\epsilon)}{N-(1+\epsilon)}$ is the so-called critical Sobolev exponent and $\ltimes_1$ is the first eigenvalue of $-\Delta_{1+\epsilon}$ given by

$$\ltimes_1 = \inf_{u \in W_0^{1,1+\epsilon}(\Psi) \setminus \{0\}} \frac{\int_\Psi |\nabla u|^{1+\epsilon}}{\int_\Psi |u|^{1+\epsilon}}.$$

Since $W_0^{1,1+\epsilon}(\Psi) \subset L(1+\epsilon)^*(\Psi)$ is a continuous but non- compact embedding, standard variational methods fail to provide solutions of (1) by minimization of the Rayleigh quotient

$$Q_\ltimes(u) = \frac{\int_\Psi |\nabla u|^{1+\epsilon} - \ltimes \int_\Psi |u|^{1+\epsilon}}{\left(\int_\Psi |u|^{(1+\epsilon)^*}\right)^{\frac{(1+\epsilon)}{(1+\epsilon)^*}}}, u \in W_0^{1,1+\epsilon}(\Psi) \setminus \{0\}.$$

Setting

$$S_\ltimes = \inf\left\{Q_\ltimes(u) : u \in W_0^{1,1+\epsilon}(\Psi) \setminus \{0\}\right\},$$

It is known that S_0 coincides with the best Sobolev constant for the embedding $\mathcal{D}^{1,1+\epsilon}(\mathbb{R}^N) \subset L^{(1+\epsilon)^*}(\mathbb{R}^N)$ and then is never attained since independent of Ψ. Moreover, by a Pohozaev identity $(1)_{\ltimes=0}$ is not solvable on star-shaped domains, see [3,14]. Sabina Angeloni, Pierpaolo Esposito [26]. The researcher intends to make few specific changes. The presence of the perturbation term $\ltimes_{u^\epsilon}$ in (1) can possibly restore compactness and produce minimizers for $Q_{\ltimes,}$ as shown for all $\ltimes > 0$ first by Brezis and Nirenberg [3] in the semi-linear case when $N \geq 4$ and then by Gunedda and Veron [14] when $N \geq 1 + 2\epsilon + \epsilon^2$.

The researcher discuss now the low-dimensional case $1 + \epsilon < N < 1 + 2\epsilon + \epsilon^2$. In the semi-linear situation $\epsilon = 1$ it corresponds to $N = 3$ and displays the following special features : according to [3]. Problem (1) is solvable on a ball precisely for $\ltimes \in \left(\frac{\ltimes_1}{4}, \ltimes_1\right)$ and then, for the minimization problem on a general domain Ψ, there holds

$$\ltimes_* = \inf\{\ltimes \in (0, \ltimes_1) : S_\ltimes < S_0\} \geq \frac{1}{4} \ltimes_1 (\mathcal{A}) = \frac{\pi^2}{4}\left(\frac{3|\Psi|}{4\pi}\right)^{-\frac{2}{3}},$$

Through a re-arrangement argument, where $\mathcal{A}$ is the ball having the same measure of Ψ. In particular, for $\ltimes \leq \frac{\ltimes_1}{4}$ a general non-existence result on $\mathcal{A}$ follows from an integral identity of Pohozacv type, obtained by testing the equation against $\psi(|y|)u'$ for a suitable smooth function ψ with $\psi(0) = 0$. An integration by parts for the term

$$\int_0^1 (\epsilon - 1)^{N-1} |u'|^{\epsilon - 1} u' u \left[\frac{\epsilon}{1+\epsilon}\psi'' - \frac{N-1}{1+\epsilon}\frac{\psi'}{-1+\epsilon} + \frac{N-1}{1+\epsilon}\frac{\psi}{(\epsilon-1)^2}\right]$$

is required to eliminate the dependence on the derivatives of u, which is possible in general just for $\epsilon = 1$. The property $\ltimes^{*} > 0$ then requires a different proof for $\epsilon \neq 1$.

Since $S_{\ltimes}$ decreases in a continuous way from S_0 to 0 as $\ltimes$ ranges in $[0, \ltimes_1[$, notice that $S_{\ltimes} = S_0$ for $0 \leq \ltimes \leq \ltimes_*$, $S_{\ltimes} < S_0$ for $\ltimes_* < \ltimes < \ltimes_1$ and $S_{\ltimes}$ is not attained for $0 \leq \ltimes < \ltimes_*$. A natural question concerns the case $\ltimes = \ltimes_*$ and the following general answer

$$S_{\ltimes_*} \text{ is not achieved} \tag{2}$$

has been given by Druet [9], with an elegant proof which unfortunately seems not to work for $\epsilon \neq 1$. A complete characterization for the critical parameter $\ltimes_*$ then follows through a blow-up approach crucially based on (2).

The researcher has use some of the results in [1]- precisely reported for reader's convenience- as a crucial ingredient to treat the quasilinear Brezis-Nirenberg problem Green function $G_{\ltimes}(\cdot, y_0)$ as a positive solution to

$$\begin{cases} -\Delta_{1+\epsilon}\, G - \ltimes G^{\epsilon} = \delta_{y_0} & \text{in } \Psi \\ G = 0 & \text{on } \partial\Psi. \end{cases} \tag{3}$$

Since uniqueness of $G_{\ltimes}(\cdot, y_0)$ is just known for $\epsilon \geq 1$, hereafter the researcher has consider the case $\epsilon > 1$. If ω_N denotes the measure of the unit ball in $\mathbb{R}^N$. Recall that the fundamental solution

$$\Gamma(y, y_0) = C_0 |y - y_0|^{-\frac{N-(1+\epsilon)}{\epsilon}}, \quad C_0 = \frac{\epsilon}{N-(1+\epsilon)}\, (N\omega_N)^{\frac{1}{\epsilon}}, \tag{4}$$

solves $-\Delta_{(1+\epsilon)}\Gamma = \delta_{y_0}$ in $\mathbb{R}^N$. The function

$$H_{\ltimes}(y, y_0) = G_{\ltimes}(y, y_0) - \Gamma(y, y_0) \tag{5}$$

is usually referred to as the " regular " part of $G_{\ltimes}(\cdot\,, y_0)$ but is just expected to be less singular than $\Gamma(y, y_0)$ at x_0.

The complete characterization in [9] for $\ltimes_*$ (see also [26] for an alternative proof) still holds in the quasi-linear case, as stated by the following main result.

Theorem 1. Let $1 \leq \epsilon < N - 1 < 1 + 2\epsilon$ and $0 < \ltimes < \ltimes_1$. The implications (i) $\Rightarrow (ii) \Rightarrow (iii)$ do hold, where
(i) there exists $y_0 \in \Psi$ such that $H_{\ltimes}(y_0, y_0) > 0$.
(ii) $S_{\ltimes} < S_0$
(iii) $S_{\ltimes}$ is attained.

Moreover, the implication (iii) $\Longrightarrow$(i) doses hold under the assumption (2) and in particular $\ltimes_{*} < 0$.

Some comments are in order. Assumption $N < 2(1+\epsilon)$ is crucial here to guarantee that $H_{\ltimes}(\cdot\,, y_0)$ is Hölder continuous at y_0, see [1]. When $2(1+\epsilon) \leq N < 1 + 2\epsilon + \epsilon^2$ we conjecture $H_{\ltimes}(y, y_0)$ to be mildly but still singular at y_0, with a behavior like $\frac{m_{\ltimes}(y_0)}{|y-y_0|^{\alpha}}$ for an appropriate $0 < \alpha < \frac{N-(1+\epsilon)}{\epsilon}$, and $m_{\ltimes}(y_0)$ to play the same role as $H_{\ltimes}(y_0, y_0)$ in the Theorem. The quantity $m_{\ltimes}(y_0)$ is usually referred to as the mass associated to $G_{\ltimes}(\cdot\,, y_0)$ and appears in several contents, see for example $[12{,}13{,}18-20]$. Notice that in the semilinear case $\epsilon = 1$ the range $2(1+\epsilon) \leq N < 1 + 2\epsilon + \epsilon^2$ is empty and such a situation doesn't see [9].

The implication (iii) $\Longrightarrow$ (i) follows by a blow –up argument once (2) is assumed.

To this aim, we first extend the pointwise blow-up theory in [10] to the quasi-linear context, a fundamental tool un the description of blow-up phenomena whose relevance goes beyond Theorem 1.1 and which completely settles some previous partial results [2,7,8] in this direction. Once sharp pointwise blow-up estimates are established, a major difficulty appears in the classical use of Pohozaev identities: written on small balls around the blow-up point as the radius tends to zero, they rule both the blow-up speed and the blow-up point location since boundary terms in such identities can be controlled thanks to the property $\nabla H_{\ltimes}(\cdot\,, y_0) \in L^{\infty}(\Psi)$. Clearly valid in the semi- linear situation, such gradient $L^{\infty}-$bound is completely missing in the quasi-linear context but surprisingly the correct answer can still be found by a different approach, based on a suitable approximation scheme for $G_{\ltimes}(\cdot\,, y_0)$. The researcher has provided a different proof of some facts in [9] in order to avoid some rough arguments concerning the limiting problems on half spaces, when dealing with boundary blow-up.

Under the assumption (2), in the proof of Theorem 1.1, the researcher will show that $H_{\ltimes_*}(y_0, y_0) = 0$ for some $y_0 \in \Psi$, a stronger property than the validity of implication $(iii) \Longrightarrow (i)$ since $H_{\ltimes}(y, y)$ is strictly increasing in $\ltimes$ for all $y \in \Psi$. Since S_0 is not attained, notice that (2) always holds if $\ltimes_* = 0$ and then $\ltimes_* > 0$ follows by the property $H_0(y_0, y_0) < 0$ for all $y_0 \in \Psi$. Moreover, since

$$\sup_{y \in \Psi} H_{\ltimes_*}(y, y) = \max_{y \in \Psi} H_{\ltimes_*}(y, y) = 0, \tag{6}$$

by monotonicity of $H_{\ltimes}$ in $\ltimes$ and under the assumption (2) the critical parameter $\ltimes_*$ is the first unique value of $\ltimes > 0$ attaining (6) and can be re –written as

$$\ltimes_* = \sup\{\ltimes \in (0, \ltimes_1) : H_{\ltimes}(y, y) < 0 \quad \text{for all } y \in \Psi\}.$$

The researcher recalls some facts from [1] that will be used throughout the paper and prove some useful convergence properties. The implication $(i) \Longrightarrow (ii)$ is established by the expansion of $Q_{\ltimes}(PU_{\epsilon,y_0})$ along the "bubble" PU_{ϵ,y_0} concentrating at y_0 as $\epsilon \longrightarrow 0$ and integral identities of Pohozaev type for $G_{\ltimes}(\cdot, y_0)$, crucial for a fine asymptotic analysis, are also derived is devoted to develop the blow-up argument along with sharp pointwise estimates to establish the final part in Theorem 1.1.

2.Some preliminary facts

For reader′s convenience, the researcher lets collect here some of the results in [26]. To give the statement of Theorem 1.1 a full meaning , the researcher has need a general theory for problem (3), as stated in the following result see [26].

Theorem 2. [1] Let $0 < \epsilon \leq N-1$ and $\ltimes < \ltimes_1$. Assume $\epsilon \geq 1$ and $N < 2 + 2\epsilon$ if $\ltimes \neq 0$. Then problem (3) has a positive solution $G_{\ltimes}(\ .\ , y_0)$ so that $H_{\ltimes}(y, y_0)$ in (5) satisfies

$$\nabla H_{\ltimes}(.\ , y_0) \in L^{\bar{q}}(\Psi), \quad \bar{q} = \frac{N(\epsilon)}{N-1}\,. \tag{7}$$

which is unique when either $\ltimes = 0$ *or* $\ltimes \neq 0$ and (7) holds. Moreover

• given $M > 0, q_0 > \frac{N}{1+\epsilon}$ and $(1+\epsilon)_0 \geq 1$ there exists $C > 0$ so that

$$\|H + c\|_{\infty, B_r(y_0)} \leq C\left(r^{-\frac{N}{(1+\epsilon)}} \quad \|H + c\|_{(1+\epsilon)_0, B_{2r}(y_0)} + r^{\frac{(1+\epsilon)q_0 - N}{q(\epsilon)}} \|f^2\|_{q_0, B_{2r}(y_0)}^{\frac{1}{\epsilon}}\right) \tag{8}$$

for all $\epsilon, 1-\epsilon,\ c \in \mathbb{R}, f^2 \in L^{q_0}(\Psi)$ and solution $G = \Gamma + H$, with $H \in L^{\infty}(\Psi)$ and $\nabla H \in L^{\bar{q}}(\Psi)$, to

$$-\nabla_{1+\epsilon}\, G + \nabla_{1+\epsilon} G = f^2 \qquad in\ \Psi \setminus \{y_0\} \tag{9}$$

so that $\epsilon^{\epsilon} \leq 1-\epsilon \leq \frac{1}{4}$ dist ($y_0, \partial\Psi$), $\frac{|y-y_0|^{\frac{1}{\epsilon}}}{M\left(\epsilon^{1+\epsilon} + |y-y_0|^{\frac{1+\epsilon}{\epsilon}}\right)^{\frac{N}{1+\epsilon}}} \leq |\nabla\Gamma| \leq M|\nabla\Gamma|(y, y_0), |c| + \|H\|_{\infty} + \|f^2\|_{q_0}^{\frac{1}{\epsilon}} \leq M$, where $\Gamma(\ .\ , y_0)$ is given by (1).

• $\ltimes\, G_{\ltimes}^{\epsilon} \in L^{q_0}(\Psi)$ for $q_0 > \frac{N}{1+\epsilon}$ *and* $H_{\ltimes}(.\ , y_0)$ is continuous function in $\bar{\Psi}$ satisfying

$$|H_{\ltimes}(y, y_0) - H_{\ltimes}(y_0, y_0)| \leq C|y - y_0|^{\alpha} \qquad \forall y \in \Psi \tag{10}$$

for some $C > 0$, $1 < \epsilon < 0$ with $H_{\ltimes}(y_0, y_0)$ strictly increasing in $\ltimes$.

Notice that the first part in Theorem 2.1 has been established in [15]. Let us stress that the condition $f^2 \in L^{q_0}(\Psi)$ for some $q_0 > \frac{N}{1+\epsilon}$, which is valid for $f^2 = \ltimes C_{\ltimes}^{\epsilon}$ when $N < 2 + 2\epsilon$ if $\ltimes \neq 0$. Of the difference equation (9) to prove $L^{\infty} -$ bounds on H as it arises for instance in the Moser iterative argument adopted in [22]. In this respect, observe that also in the semilinear case $H_{\ltimes}(. \, , y_0)$ is no longer regular at y_0 when $2 = (1+\epsilon) \leq \frac{N}{2}$.

The following a-priori estimates are the basis of Theorem 2.1 and will be crucially used here to establish some accurate pointwise blow-up estimates.

Proposition 1. [**1**] Let $1 \leq \epsilon \leq N-1$. Assume that $(1-\epsilon)_n \in L^{\infty}(\overline{\Psi})$ $f_{(1-\epsilon)}^2 \in L^1(\Psi)$ and $g_{(1-\epsilon)}^2$, $g_{(1-\epsilon)}^2$ satisfy. $g_{1-\epsilon}^2$, $\hat{g}_{1-\epsilon}^2 \in L^{\infty}(\Psi) \cap W^{1,1+\epsilon}(\Psi)$(1+$\epsilon$)-harmonic in Ψ, $g_{1-\epsilon}^2$, $\hat{g}_{1-\epsilon}^2$ non-constant unless 0 and $\lim_{\epsilon \to \infty} \|(1+\epsilon)_{\epsilon} - (1+\epsilon)\|_{\infty} = 0$ with $\sup_{\Psi}(1+\epsilon) < \ltimes_1$, $\sup_{\epsilon \in \mathbb{N}} [\|f_{\epsilon}^2\|_1 + \|g_{\epsilon}^2\|_{\infty} + \|\hat{g}_{\epsilon}^2\|_{\infty}] < +\infty$.

If $u_{1-\epsilon} \in W_{g_{1-\epsilon}}^{1,1+\epsilon}(\Psi)$ solves $-\Delta_{1+\epsilon} u_{1-\epsilon} - u_{1-\epsilon}|u_{1-\epsilon}|^{\epsilon+3}\, \hat{u}_{1-\epsilon} = f^2{}_{1-\epsilon} \; in \; \Psi$, then $\sup_{(1+\epsilon) \in \mathbb{N}} \|u_{1-\epsilon} - \hat{u}_{1-\epsilon}\|_{\infty} < \infty >$.

Proposition 2. Let $1 \leq \epsilon \leq N-1$ and 1+ϵ ,$f_1^2, f_2^2 \in L^{\infty}(\Psi)$. Let $u_i \in C^1(\overline{\Psi})$, $i = 1{,}2, be$ solutions to

$$-\Delta_{1+\epsilon}(1+\epsilon)_{u_i} - (1+\epsilon)u_i^{\epsilon} = f_i^2 \qquad in\ \Psi$$

so that

$$u_i > 0 \;\; in\ \Psi, \quad \frac{u_1}{u_2} \leq C \;\; near \;\; \partial\Psi$$

for some $C > 0$. $If \;\; f_1^2 \leq \; f_2^2 \;\; with \;\; f_2^2 \geq 0 \; in\ \Psi \;\; and\ u_1 \leq u_2 \;\; on\ \partial\Psi, then\ u_1 \leq u_2 \; in\ \Psi$.

The researcher has introduce now a special approximation scheme for $C_{\ltimes}(. \, , y_0)$. Given $C_1 = N^{\frac{N-(1+\epsilon)}{(1+\epsilon)^2}} \left(\frac{N-(1+\epsilon)}{\epsilon}\right)^{\frac{\epsilon(N-(1+\epsilon)}{1+2\epsilon+\epsilon^2}}$, the so- called standard bubbles

$$U_{(\epsilon\, , y_0)}(y) = C_1 \left(\frac{\epsilon}{\epsilon^{1+\epsilon} + |y - y_0|^{\frac{1+\epsilon}{\epsilon}}} \right)^{\frac{N-(1+\epsilon)}{\epsilon}} \qquad \epsilon > 0\, , y_0 \in \mathbb{R}^N, \tag{11}$$

Are the extremals of the Sobolev inequality

$$S_0 \left(\int_{\mathbb{R}N} |u|^{(1+\epsilon)^*} \right)^{\frac{1+\epsilon}{(1+\epsilon)^*}} \leq \int_{\mathbb{R}N} |\nabla u|^{1+\epsilon}, \quad u \in \mathcal{D}^{1,1+\epsilon}(\mathbb{R}^N),$$

and the unique solution in $\mathcal{D}^{1,1+\epsilon}(\mathbb{R}^N)$ of

$$-\Delta_{1+\epsilon} U = U^{(1+\epsilon)^*-1} \quad \text{in } \mathbb{R}^N, \tag{12}$$

see [5,21,25]. For $\ltimes < \ltimes_1$ consider its projection PU_{ϵ,y_0} in Ψ, as the solution of

$$\begin{cases} -\nabla_{1+\epsilon} PU_{\epsilon,y_0} = \ltimes PU_{\epsilon,y_0}^{\epsilon} + U_{\epsilon,y_0}^{\epsilon^*} & in\ \Psi \\ PU_{\epsilon,y_0} > 0 & in\ \Psi \\ PU_{\epsilon,y_0} = 0 & on\ \partial\Psi \end{cases} \tag{13}$$

The researcher lets $G_{\epsilon,y_0} = \frac{C_0}{C_1}\epsilon^{-\frac{N-1-\epsilon}{1+\epsilon}} PU_{\epsilon,y_0}$ with given by (4), decompose it as $G_{\epsilon,y_0} = \Gamma_{\epsilon,y_0} + H_{\epsilon,y_0}$, where

$$\Gamma_{\epsilon,y_0} = \frac{C_0}{C_1}\epsilon^{-\frac{N-(1+\epsilon)}{1+\epsilon}} U_{\epsilon,y_0} = \frac{C_0}{\left(\epsilon^{1+\epsilon} + |y-y_0|^{\frac{1+\epsilon}{\epsilon}}\right)^{\frac{N-(1+\epsilon)}{1+\epsilon}}} \to \Gamma(y,y_0) \tag{14}$$

in $C_{loc}^1\ (\overline{\Psi} \setminus \{y_0\})$ as $\epsilon \to 0$. Since

$$f_{\epsilon,y_0}^2 = -\nabla_{1+\epsilon}\,\Gamma_{\epsilon,y_0} = \frac{C_0}{C_1}\epsilon^{-\frac{N-(1+\epsilon)}{1+\epsilon}} U_{\epsilon,y_0}^{\epsilon^*} \quad \frac{C_0^{\epsilon} C_1^{\frac{1+2\epsilon+\epsilon^2}{N-(1+\epsilon)}} \epsilon^{1+\epsilon}}{\left(\epsilon^{1+\epsilon} + |y-y_0|^{\frac{1+\epsilon}{\epsilon}}\right)^{\frac{N-(1+\epsilon)}{1+\epsilon}}} \to 0 \tag{15}$$

in $C_{loc}(\overline{\Psi} \setminus \{y_0\})$ and

$$\int_{\Psi} f_{\epsilon,y_0}^2 = \int_{\partial\Psi} \left|\nabla\Gamma_{\epsilon,y_0}\right|^{\epsilon-1} \partial_v \Gamma_{\epsilon,y_0} \to -\int_{\partial\Psi} |\nabla\Gamma|^{\epsilon-1}(y,y_0)\,\partial_v\,\Gamma(y,y_0)(1+\epsilon)\sigma \ -1 = 0$$

as $\epsilon \to 0$ in view of (12) and (14) notice that $f_{\epsilon,y_0}^2 \to \delta_{y_0}$ weakly in the sense of measures in Ψ as $\epsilon \to 0$ and $G_{\epsilon,\,y_0}$ solves

$$\begin{cases} -\Delta_{1+\epsilon} G_{\epsilon,y_0} = \ltimes G_{\epsilon,y_0}^{\epsilon} + f_{\epsilon,y_0}^2 & \text{in } \Psi \\ G_{\epsilon,y_0} > 0 & \text{in } \Psi \\ G_{\epsilon,y_0} = 0 & \text{on } \partial\Psi \end{cases} \tag{16}$$

Proposition 3. Let $1 \le \epsilon \le N-1$ and assume $N < 2+2\epsilon$ if $\ltimes \neq 0$ .Then there holds

$$H_{\epsilon,y_0} \longrightarrow H_{\ltimes}(.\ ,y_0) \qquad in\ C(\Psi) \tag{17}$$

at $\epsilon \longrightarrow 0$.

Proof. By proposition 1 the researcher cans find a subsequence $\epsilon_n \longrightarrow 0$ so that $G_{\epsilon_n,y_0} \longrightarrow G\ in\ W_0^{1,1+2\epsilon}(\Psi)\ \ at\ \epsilon \longrightarrow +\infty$ for all $0 \le \epsilon < \bar{\epsilon}$, where $G = \Gamma(y,y_0) + H$ is a solution of (3) for some H in view of (14) and (16) if $\ltimes \neq 0$ by the Sobolev embedding theorem there holds

$$G_{\epsilon_n,y_0} \longrightarrow G \qquad \text{in } L^{1+\epsilon}(\Psi) \qquad \text{at } \epsilon \longrightarrow +\infty \tag{18}$$

the researcher lets $\epsilon > 0$ in view of $N < 2+2\epsilon \le 1+2\epsilon+\epsilon^2$ rewrite (16) in the equivalent form :

$$\begin{cases} -\Delta_{1+\epsilon}\left(\Gamma_{\epsilon,y_0} + H_{\epsilon,y_0}\right) + \Delta_{1+\epsilon}\Gamma_{\epsilon,y_0} = \ltimes\, G^{\epsilon}_{\epsilon,y_0} & in\ \Psi \\ H_{\epsilon,y_0} = -\Gamma_{\epsilon,y_0} & on\ \ \partial\Psi \end{cases} \tag{19}$$

the researcher lets denote the solution of $(16)_{\ltimes=0}$ by $\mathrm{C}^0_{\epsilon,y_0}$ and set $\mathrm{H}^0_{\epsilon,y_0} + \Gamma_{\epsilon,y_0}$
$= \mathrm{C}^0_{\epsilon,y_0}$. By the uniqueness part in Theorem
$= 0$ the researcher has 2 with $\ltimes$

$$C^0_{\epsilon,y_0} \longrightarrow G_0(.\ ,y_0)\ \ in\ \ W_0^{1,1+2\epsilon}(\Psi)$$

$at\ \epsilon \longrightarrow 0, for\ all\ \ 0 \le \epsilon < \bar{\epsilon}. Moreover$, since $\left|H^0_{\epsilon,y_0}\right| \le M$ on $\partial\Psi$, by integrating (19) against $(H^0_{\epsilon,y_0} \mp M)$ the researcher denotes

$$\left|H^0_{\epsilon,y_0}\right| \le M \qquad in\ \ \Psi \tag{20}$$

In an uniform ways and then $\mathrm{C}^0_{\epsilon,y_0}$ is locally uniformly bounded in $\bar{\Psi} \setminus \{y_0\}$. By elliptic estimates [6,16,22,23] and $(16)_{\ltimes=0}$ the researcher denotes that

$$G^0_{\epsilon,y_0} \text{ uniformly bounded in } C_{loc}^{1,\frac{1}{2}+\epsilon}(\bar{\Psi} \setminus \{y_0\})\ \ , \tag{21}$$

For some $-\frac{1}{2} < \alpha < \frac{1}{2}$. Integrating $(19)_{\ltimes=0}$ against $\eta^{1+\epsilon}\ H^0_{\epsilon,y_0}$, $0 \le \eta \in C_0^\infty(\Psi)$, the researcher gets that

$$\int_{\Psi} \eta^{1+\epsilon}\left|\nabla H^0_{\epsilon,y_0}\right|^{1+\epsilon} \le (1+\epsilon)\int_{\Psi} \eta^{\epsilon}\,|\nabla\eta|(\left|\nabla\Gamma_{\epsilon,y_0}\right|^{\epsilon-1}+\left|\nabla H^0_{\epsilon,y_0}\right|^{\epsilon-1})\left|H^0_{\epsilon,y_0}\right|\left|\nabla H^0_{\epsilon,y_0}\right|$$

and then (20) Young′s inequality imply that

$$\nabla H^0_{\epsilon,y_0} \text{ uniformly bounded in } L^{1+\epsilon}\,(\Psi) \tag{22}$$

in view of (21) the researcher considers now the case $\ltimes\neq 0$. Since

$$-\Delta_{1+\epsilon}\left(\Gamma_{\epsilon,y_0}+H_{\epsilon,y_0}\right)+\Delta_{1+\epsilon}\left(\Gamma_{\epsilon,y_0}+H^0_{\epsilon,y_0}\right)=\ltimes\, G^{\epsilon}_{\epsilon,y_0} \qquad \text{in } \Psi$$

with $H^0_{\epsilon,y_0}=H_{\epsilon,y_0}$ on $\partial\Psi$, an integration against $H_{\epsilon,y_0}-H^0_{\epsilon,y_0}=0$ gives that

$$\int_{\Psi}\left|\nabla(H_{\epsilon,y_0}-H^0_{\epsilon,y_0})\right|^{1+\epsilon} \le |\ltimes|\int_{\Psi} G^{\epsilon}_{\epsilon,y_0}\left|H_{\epsilon,y_0}-H^0_{\epsilon,y_0}\right| \le |\ltimes|\left\|G_{\epsilon,y_0}\right\|^{\epsilon}_{1+\epsilon}\left\|H_{\epsilon,y_0}-H^0_{\epsilon,y_0}\right\|_{1+\epsilon}$$

gratitudes to the Hölder′s inequality and the coercivity properties of the (1+ϵ) $-$ Laplace operator, and then

$$\nabla\left(H_{\epsilon_n,y_0}-H^0_{\epsilon_n,y_0}\right) \text{ uniformly bounded in } L^{1+\epsilon}\,(\Psi) \tag{23}$$

in view of (18) and Poincar′s inequality. A combination of (22) and (23) lead to a uniform $L^{1+\epsilon}$ $-$ bound on $\nabla H^0_{\epsilon_n,y_0}$, showing by Fatou′s lemma that $\nabla H\in L^{1+\epsilon}$ (Ψ) . By Theorem 2 , the researcher has that $G=G_{\ltimes}(\,.\;,y_0)$ and then

$$G_{\epsilon,y_0}\longrightarrow G_{\ltimes}(.\;,y_0) \text{ in } W_0^{1,1+2\epsilon}\,(\Psi) \tag{24}$$

at $\epsilon\longrightarrow 0$, for all $0\le\epsilon<\bar{\epsilon}$.

To extend (20) to the case $\ltimes\neq 0$, observe that (16) and $-\Delta_{1+\epsilon}\Gamma_{\epsilon,y_0}=f^2_{\epsilon,y_0}$ in Ψ imply $\left\|H_{\epsilon,y_0}\right\|_{\infty}\le C$ for all $\epsilon>0$ thanks to proposition 1 in view of $N<2+2\epsilon$ when $\ltimes\neq 0$. Since $f^2=\ltimes\, G^{\epsilon}_{\epsilon,y_0}$ is uniformly bounded in $L^{(1+2\epsilon)0}$ (Ψ) for some (1+2ϵ)0 $>\frac{N}{1+\epsilon}$ in view of $\frac{\overline{1+2\epsilon}}{\epsilon}>\frac{N}{1+\epsilon}$ when $N<2+2\epsilon$ and

$$\left|\nabla\Gamma_{\epsilon,y_0}\right|=\frac{C_0(N-(1+\epsilon)}{\epsilon}\;\frac{|y-y_0|^{\frac{1}{\epsilon}}}{\left(\epsilon^{1+\epsilon}+|y-y_0|^{\frac{1}{\epsilon}}\right)^{\frac{N}{1+\epsilon}}}\le M|\nabla|(y,y_0),$$

we can apply (8) in Theorem 2 to H_{ϵ,y_0} as a solution to (19) by getting

$$\left|H_{\epsilon,y_0}(y) - H_{\ltimes}(y_0, y_0)\right| \leq C\left(r^{-\frac{N}{\epsilon}} \left\|H_{\epsilon,y_0}(y) - H_{\ltimes}(y_0, y_0)\right\|_{\epsilon}, B_{2r}(y_0) + r^{\frac{(1+\epsilon)(1+2\epsilon)-N}{(1+2\epsilon)0(\epsilon)}}\right) \quad (25)$$

for all $y \in B_R(y_0)$ and $1 - \epsilon^{\epsilon} \leq \epsilon \leq \frac{3}{4}$ dist $(y_0, \partial\Psi)$.

By contradiction assume that (17) does not hold. Then there exist sequences $\epsilon_n \longrightarrow 0$ and $y_0 \in \Psi$ so that $\left|H_{\epsilon_n,y_0}(y_n) - H_{\ltimes}(y_n, y_0)\right| \geq 2\delta > 0$. Since by elliptic estimates [6,16,22,23,26] there holds

$$G_{\epsilon,y_0} \longrightarrow G_{\ltimes}(.\ , y_0) \text{ in } C^1_{loc}(\overline{\Psi} \setminus \{y_0\} \quad (26)$$

at $\epsilon \longrightarrow 0$ in view of (16) and (24), the researcher has that $\bar{y} - y_0 = 0$ and then

$$\left|H_{\epsilon_n,y_0}(y_n) - H_{\ltimes}(y_n, y_0)\right| \geq \delta \quad (27)$$

thanks to $H_{\ltimes}(.\ , y_0) \in C(\overline{\Psi})$. Since by the Sobolev embedding theorem $H_{\epsilon,y_0} \longrightarrow H_{\ltimes}(.\ , y_0)$ in $L^{\epsilon}(\Psi)$ at $\epsilon \longrightarrow 0$ in view of (24) and $\overline{1 + 2\epsilon} > \epsilon$, we can insert (27) into (25) and get $\epsilon \longrightarrow +\infty$

$$\partial \leq C\left(r^{-\frac{N}{\epsilon}} \|H_{\ltimes}(., y_0) - H_{\ltimes}(y_0, y_0)\|_{\epsilon}, B_{2r}(y_0) + r^{\frac{1+\epsilon(1+2\epsilon)0-N}{(1+2\epsilon)0(\epsilon)}}\right) \quad (28)$$

for all $\frac{1}{2} > \epsilon \geq \frac{1}{4}$ dist $(y_0, \partial\Psi)$. Since

$$r^{-\frac{N}{\epsilon}} \|H_{\ltimes}(., y_0) - H_{\ltimes}(y_0, y_0)\|_{\epsilon}, B_{2r}(y_0) \leq Cr^{\propto} \longrightarrow 0$$

at $\epsilon \longrightarrow \frac{1}{2}$ thanks to (10), estimate (28) leads to a contradiction and the proof is complete.

Corollary 1. Let $1 \leq \epsilon \leq N - 1$ and assume $N < 2 + 2\epsilon$ if $\ltimes \neq 0$. Then the expansion

$$PU_{\epsilon,y_0} = U_{\epsilon,y_0} + \frac{C_1}{C_2}\, \epsilon^{\frac{N-(1+\epsilon)}{1+\epsilon}}\, H_{\ltimes}(.\ , y_0) + O\left(\epsilon^{\frac{N-(1+\epsilon)}{1+\epsilon}}\right) \quad (29)$$

does hold uniformly in Ψ at $\epsilon \longrightarrow 0$.

3. Energy expansions and Pohozaev identities

The researcher concerned with the discussion of implication $(i) \Rightarrow (ii)$ in Theorem 1, whereas the proof of $(ii) \Rightarrow (iii)$ in Theorem 1 is rather classical and can be found in [14].

Let $0 < \ltimes < \ltimes_1$ and $y_0 \in \Psi$ so that $H_{\ltimes}(y_0, y_0) > 0$. The researcher shows $S_{\ltimes} < S_0$ let expand $H_{\ltimes}(PU_{\epsilon,y_0})$ for $\epsilon > 0$ small. Since PU_{ϵ,y_0} solves (13), the researcher that

$$\int_{\Psi} \left|\nabla PU_{\epsilon,y_0}\right|^{1+\epsilon} - \ltimes \int_{\Psi} \left(PU_{\epsilon,y_0}\right)^{1+\epsilon} = \int_{\Psi} U_{\epsilon,y_0}^{(1+\epsilon)^*-1} PU_{\epsilon,y_0} = \int_{\Psi} U_{\epsilon,y_0}^{(1+\epsilon)^*}$$

$$+ \frac{C_1}{C_2} \epsilon^{\frac{N-(1+\epsilon)}{1+\epsilon}} \int_{\Psi} U_{\epsilon,y_0}^{(1+\epsilon)^*-1} \left[H_{\ltimes}(y, y_0) + o(1)\right] \quad (30)$$

at $\epsilon \longrightarrow 0$ in view of (29). Given $\Psi_\epsilon = \frac{\Psi - y_0}{\epsilon^\epsilon}$ observe that

$$\int_{\Psi} U_{\epsilon,y_0}^{(1+\epsilon)^*} = \int_{\Psi_\epsilon} U_1^{(1+\epsilon)^*} = \int_{\mathbb{R}^N} U_1^{(1+\epsilon)^*} + O(\epsilon^N) \qquad (31)$$

And

$$\int_{\Psi} U_{\epsilon,y_0}^{(1+\epsilon)^*-1} \left[H_{\ltimes}(y, y_0) + o(1)\right] = \int_{\Psi} U_{\epsilon,y_0}^{(1+\epsilon)^*-1} \left[H_{\ltimes}(y, y_0) + O(|y - y_0|)^\alpha + o(1)\right]$$

$$= \epsilon^{\frac{(N-(1+\epsilon)(\epsilon)}{1+\epsilon}} \int_{\Psi_\epsilon} U_1^{(1+\epsilon)^*-1} \left[H_{\ltimes}(y, y_0) + O|\epsilon^{\alpha\epsilon}|^\alpha\, y + o(1)\right]$$

$$= \epsilon^{\frac{(N-(1+\epsilon)(\epsilon)}{1+\epsilon}} H_{\ltimes}(y, y_0) \int_{\mathbb{R}^N} U_1^{(1+\epsilon)^*} + O\left(\epsilon^{\frac{(N-(1+\epsilon)(\epsilon)}{1+\epsilon}}\right) \qquad (32)$$

In view (10) and $\int_{\mathbb{R}^n} U_1^{(1+\epsilon)^*-1} |y|^\alpha < +\infty$. Inserting (31)-(32) into (31) the researcher deduces

$\int_\Psi \left|\nabla PU_{\epsilon,y_0}\right|^{1+\epsilon} - \ltimes \int_\Psi \left(PU_{\epsilon,y_0}\right)^{1+\epsilon} = \int_{\mathbb{R}^N} U_1^{(1+\epsilon)^*} +$
$\epsilon^{N-(1+\epsilon)} \frac{C_1}{C_0} H_\ltimes(y_0, y_0) \int_{\mathbb{R}^N} U_1^{(1+\epsilon)^*} + o\left(\epsilon^{N-(1+\epsilon)}\right).$
(33)
By the Taylor expansion

$$\left(PU_{\epsilon,y_0}\right)^{(1+\epsilon)^*} = U_{\epsilon,y_0}^{(1+\epsilon)^*} + \epsilon^{\frac{N-(1+\epsilon)}{1+\epsilon}} \frac{C_1}{C_2} (1+\epsilon)^* U_{\epsilon,y_0}^{(1+\epsilon)^*-1} [H_\ltimes(y, y_0) + o(1)] + O\epsilon^{2\frac{N-(1+\epsilon)}{1+\epsilon}} U_{\epsilon,y_0}^{(1+\epsilon)^*-2} + \epsilon^N$$

in view of (29) and $\|H_\ltimes(.,y_0)\|_\infty < +\infty$, the researcher obtains

$$\int_\Psi \left(PU_{\epsilon,y_0}\right)^{(1+\epsilon)^*} = \int_{\mathbb{R}^N} U_1^{(1+\epsilon)^*} + \epsilon^{N-(1+\epsilon)} \frac{C_1}{C_2} (1+\epsilon)^* H_\ltimes(y_0, y_0) \int_{\mathbb{R}^N} U_{\epsilon,y_0}^{(1+\epsilon)^*-1} + o\left(\epsilon^{N-(1+\epsilon)}\right) \qquad (34)$$

thanks to (33)-(34) and

$$\int_\Psi U_{\epsilon,y_0}^{(1+\epsilon)^*-2} = \epsilon^{2\frac{N-(1+\epsilon)(\epsilon)}{1+\epsilon}} \int_{\Psi_\epsilon} U_1^{(1+\epsilon)^*-2} = O\left(\epsilon^{2\frac{N-(1+\epsilon)(\epsilon)}{1+\epsilon}}\right)$$

for $N < 2 + 2\epsilon$. Expansions (33)-(34) .

4.The amplification approach

Following [9] let us introduce the following blow –up procedure. Letting $\ltimes_{1-\epsilon} = \ltimes_\star + \frac{1}{1-\epsilon}$, the researcher has $S_{\ltimes_{1-\epsilon}} < S_0 = S_{\ltimes_\star}$ and then $S_{\ltimes_{1+\epsilon}}$ is achieved by a nonnegative $u_{1+\epsilon} \in W_0^{1,1+\epsilon}(\Psi)$ which , up to a normalization, satisfies

$$-\Delta_{1-\epsilon} u_{1-\epsilon} = \ltimes_{1-\epsilon} u_{1-\epsilon}^{\epsilon} + u_{1-\epsilon}^{(1+\epsilon)^*-1} \text{ in } \Psi, \int_\Psi u_{1-\epsilon}^{(1+\epsilon)^*} = S_{\ltimes_{1-\epsilon}}^{\frac{N}{1+\epsilon}}. \qquad (35)$$

Since $\ltimes_* < \ltimes_1$, by (35) the sequence $u_{1-\epsilon}$ is uniformly bounded in $W_0^{1,1+\epsilon}(\Psi)$ and then, up to a sub sequence $u_{1-\epsilon} \longrightarrow u_0 \geq 0$ in $W_0^{1,1+\epsilon}(\Psi)$. In Ψ as $\epsilon \longrightarrow +\infty$. Since

$$Q_{\ltimes_{1-\epsilon}}(u) = Q_{\ltimes_*}(u) - \frac{1}{1-\epsilon} \frac{\|u_{1-\epsilon}\|_{1+\epsilon}^{1+\epsilon}}{\|u_{1-\epsilon}\|_{(1+\epsilon)^*}^{1+\epsilon}} \geq S_0 - \frac{C}{1-\epsilon}$$

for some $C > 0$ thanks to the Hölder′s inequality, the researcher deduces that

$$\lim_{\epsilon \longrightarrow +\infty} S_{\ltimes_{1-\epsilon}} = S_0. \tag{36}$$

By letting $\epsilon \longrightarrow +\infty$ in (35) the researcher deduces that $u_0 \in W_0^{1,1+\epsilon}(\Psi)$. Solves

$$-\Delta_{1-\epsilon} u_0 = \ltimes_0 u_{1-\epsilon}^{\epsilon} + u_0^{(1+\epsilon)^*-1} \quad \text{in} \quad \Psi, \int_\Psi u_0^{(1+\epsilon)^*} \leq S_{\ltimes_0}^{\frac{N}{1+\epsilon}},$$

thanks to $u_{1-\epsilon} \longrightarrow u_0$ in Ψ as $\epsilon \longrightarrow 0$ and the Fatou convergence Theorem, and then

$$S_0 \leq Q_{\ltimes_*}(u_0) = \left(\int_\Psi u_0^{(1+\epsilon)^*} \right)^{\frac{1+\epsilon}{N}} \leq S_0$$

if $u_0 \neq 0$. Since $S_{\ltimes_*} = S_0$ would be achieved by u_0 if $u_0 \neq 0$, assumption (2) is crucial to guarantee $u_0 = 0$ and then

$$u_{1-\epsilon} \longrightarrow 0 \text{ in } W_0^{1,1+\epsilon}(\Psi). \;\; u_{1-\epsilon} \longrightarrow 0 \qquad L^{1+2\epsilon}(\Psi) \;\; \text{for } 1 \leq 1+2\epsilon < (1+\epsilon)^* \;\; \text{in } \Psi \tag{37}$$

in view of the Soblev embedding Theorem.

5- Results and Recommendations

1- Assume by contradiction $y_0 \in \partial\Psi$ and set $\hat{y} - y_0 + \nu(y_0) = 0$. Let us apply the Pohozaev identity to $u_{1-\epsilon}$ with $\epsilon = 1$, $f^2 = 0$ and $y_0 - \hat{y} = 0$ on $D = \Psi$, to get

$$\int_{\partial\Psi} |\nabla u_{1-\epsilon}|^{1+\epsilon} \langle y - \hat{y}, \nu \rangle = \frac{1+\epsilon}{\epsilon} \ltimes_{1-\epsilon} \int_\Psi u_{1-\epsilon}^{1+\epsilon} \tag{38}$$

2 − I recommend end up with contradictory conclusion $\nabla G = 0$ on $\partial\Psi_\infty$, and then $\notin \partial\Psi$. There holds $H_{\ltimes_*}(y_0, y_0) = 0$.
Let us apply the Pohozaev identity to $u_{1-\epsilon}$ with $\epsilon = 1$, $f^2 = 0$ on $D = \mathcal{B}_\delta(y_0) \subset \Psi$ and to get

$$\ltimes_{1-\epsilon} \int_{\mathcal{B}_\delta(y_0)} u_{1-\epsilon}^{1+\epsilon} + \int_{\partial\mathcal{B}_\delta(y_0)} \frac{\delta}{1+\epsilon} |\nabla u_{1-\epsilon}|^{1+\epsilon} - \delta |\nabla u_{1-\epsilon}|^{\epsilon-1} (\partial_\nu u_{1-\epsilon})^2$$
$$- \frac{\ltimes_{(1-\epsilon)}\, \delta}{1+\epsilon} u_{1-\epsilon}^{1+\epsilon} - \frac{N-(1+\epsilon)}{N(1+\epsilon)} u_{1-\epsilon} |\nabla u_{1-\epsilon}|^{\epsilon-1} \partial_\nu u_{1-\epsilon}$$
$$- \frac{N-(1+\epsilon)}{N(1+\epsilon)} \int_{\partial\mathcal{B}_\delta(y_0)} u_{1-\epsilon}^{(1+\epsilon)^*} = 0 \qquad (38)$$

References

[1] S. Angeloni, P. Esposito, The Green function for $(1+\epsilon) -$ Laplace operation, preprint. Ariv:2203.01206.
[2] T. Aubin. Y. Li, On the best Sobolev inequality. J . Math. Purres Appl. (9) 78 (4) (1999) 353-387.
[3] H. Brezis, L. Nirenberg, positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Commun. Pure Apple. Math. 36 (1983) 437-477.
[4] L.Damascelli, A. Farina, B. Seiunzi, E. Valdinoei, Liouville, results for m-Laplace equations of Lame –Emden –Fower type, Ann. Inst. Henri Poincarè, Anal. Non Linèaire 26 (4)(2009) 1099-1119.
[5] E. Dibenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (8) (1983) 827-850.
[6] Z. Diadli, O. Druet, Extremal functions for optimal Sobolev inequalities on compact manifolds. Cale. Var. Partial Differ. Eqn. 12 (1) (2001) 59-84.
[7] O. Druet. The best constants problem in Sobolev inequalities. Math. Ann. 314 (2) (1999) 327-346.
[8] O. Druet. Elliptic equations with critical Sobolev exponents in dimension 3. Ann. Inst. Henri Poincarè. Anal. Non Linèaire 19 (2) (2002) 125-112.
[9] O. Druet. F. Robert, The Hebey, Blow-Up Theory for Elliptic PDEs in Riemannian Ceometry. Mathematical Notes, vol. 45, Princeton University Press, Princeton, NJ,2004.
[10] P. Esposito, On some conjectures proposed by Haïm Brezis, Nonlinear Anal. 54 (5) (2004) 751-751.
[11] N. Ghonssonb, F. Robert, Hardy- singular boundary mass and Sobolev-critical variational problems, Anal. PDE 10 (5) (2017) 1017 – 1079.

[12] N. Ghonssonb, F. Robert, The Hardy-Schödinger operator with interior singularity: the remaining cases, Cale. Var. Partial Differ. Egu. 56 (5) (2017) 149.
[13] M. Gusdda, L. Veron, Quasilinear elliptic equations involving critical Sobolev exponents. Nonlinear Anal. 13 (8) (1989) 879-902.
[14] S. Kichenassamy. L. Veron, Singular solutions of the $(1+\epsilon)-Laplace$ equations, Math Ann. 275 (1986) 599-616.
[15] G.M. Lieberman, Boundary regularity for solution of degenerate elliptic equations, Nolinear Anal 12(11) (1988) 1203-1219.
[16] R. Sehoen. Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differ Geom. 20 (2) (1984) 479-495.
[17] R. Sehoen. S.T. Yau, Proof of the positive mass theorem. H, Commun. Math. Phys. 79 (2) (1981) 231-260.
[18] B. Sciunzi, Classification of positive $\mathcal{D}^{1,1+\epsilon}$ ($\mathbb{R}^N$)-solution to the critical P-Laplace equation in $\mathbb{R}^N$, Adv. Math. 291 291 (2016) 12-23.
[19] I. Serrin, Local behavior of solutions of quasilinear equations, Acta Math. 111 (1964) 217-302.
[20] P. Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. J. Differ. Equ 51 (1984) 126-150.
[21] J.I. Vazquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math Optim. 12 (1984) 191- 202.
[22] I. Vetois, A priori estimates and application to the symmetry for critical (1+ϵ)-Laplace equations. J. Differ. Equ. 260 (1) (2016) 149- 161.
[23] Sabina Angeloni, Pierpaolo Esposito, The quasi-linear Brezis-Nirenberg problem in Low dimensions, Journal of functional Analysis. 286 (2024) 110176.