Application of Hardy Identities and Inequalities on Cartan-Hadamard Manifolds

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Abstract

We study the Hardy identities and inequalities on Cartan-Hadamard manifolds using the notion of a Bessel pair. These Hardy identities offer significantly more information on the existence/nonexistence of the extremal functions of the Hardy inequalities. These Hardy inequalities are in the spirit of Brezis-Vázquez in the Euclidean spaces. As an application on the way of [43], we establish several Hardy type inequalities that show improvements as well as simple understandings to many known Hardy inequalities and Hardy-Poincaré-Sobolev type inequalities on hyperbolic spaces in the literature, with a little bite of change.

Keywords and phrases. Hardy inequality; Hardy-Poincaré-Sobolev; Cartan-Hadamard manifold; Hyperbolic space.

1. Introduction

We study (see [43]) the improvements of the L^2-Hardy type inequalities on Cartan-Hadamard manifold, i.e. a Riemannian manifold that is complete and simply connected and has everywhere nonpositive sectional curvature. We also sharpen Hardy inequalities on hyperbolic spaces.

On the Euclidean space $\mathbb{R}^{3+\epsilon}, \epsilon \geq 0$, the following Hardy inequality plays important roles in many areas:

$$\int_{\mathbb{R}^{3+\epsilon}} \sum |\nabla f_j|^2 dx \geq \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{R}^{3+\epsilon}} \sum \frac{|f_j|^2}{|x|^2} dx, \quad f_j \in C_0^\infty(\mathbb{R}^{3+\epsilon}). \qquad (1.1)$$

The constant $\left(\frac{1+\epsilon}{2}\right)^2$ in (1.1) is optimal. Therefore, we improve (1.1) by adding extra nonnegative terms to its right hand side. On $\mathbb{R}^{3+\epsilon}$, the operator $-\Delta - \left(\frac{1+\epsilon}{2}\right)^2 \frac{1}{|x|^2}$ is known to be critical and there is no strictly positive functions $V_j \in V_j^1((0,\infty))$ such that the inequality

$$\int_{\mathbb{R}^{3+\epsilon}} \sum |\nabla f_j|^2 dx - \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{R}^{3+\epsilon}} \sum \frac{|f_j|^2}{|x|^2} dx \geq \int_{\mathbb{R}^{3+\epsilon}} \sum V_j(|x|)|f_j|^2 dx$$

holds for all $f_j \in C_0^\infty(\mathbb{R}^{3+\epsilon})$ (see [26, Corollary 2.3.4]). The situation is very different on bounded domains. It has been showed that extra nonnegative terms can be added to the Hardy inequality on bounded domains. For Ω be a bounded domain in $\mathbb{R}^{3+\epsilon}, \epsilon \geq 0$, with $0 \in \Omega$, we investigate the stability of singular solutions of nonlinear elliptic equations, Brezis and Vázquez verified in [12] that for all $f_j \in (W_j)_0^{1,2}(\Omega)$:

$$\int_\Omega \sum |\nabla f_j|^2 dx - \left(\frac{1+\epsilon}{2}\right)^2 \int_\Omega \sum \frac{|f_j|^2}{|x|^2} dx \geq z_0^2 \omega_{3+\epsilon}^{\frac{2}{3+\epsilon}} |\Omega|^{-\frac{2}{3+\epsilon}} \int_\Omega \sum |f_j|^2 dx \qquad (1.2)$$

where $\omega_{3+\epsilon}$ is the volume of the unit ball and $z_0 = 2.4048\ldots$ is the first zero of the Bessel function $J_0(z)$. We also mention that in [42], Vázquez and Zuazua established the following improved Hardy-Poincaré inequality: for any $0 \leq \epsilon < 2$, there exists a constant $(1+\epsilon)(1+\epsilon,\Omega) > 0$ such that for all $f_j \in (W_j)_0^{1,2}(\Omega)$:

$$\int_\Omega \sum |\nabla f_j|^2 dx - \left(\frac{1+\epsilon}{2}\right)^2 \int_\Omega \sum \frac{|f_j|^2}{|x|^2} dx \geq (1+\epsilon)(1+\epsilon,\Omega) \| f_j \|_{W^{1,1+\epsilon}(\Omega)}^2$$

Note that the constant $z_0^2 \omega_{3+\epsilon}^{\frac{2}{3+\epsilon}} |\Omega|^{-\frac{2}{3+\epsilon}}$ in (1.2) is optimal when Ω is a ball and again is not attained in $(W_j)_0^{1,2}(\Omega)$. Therefore, Brezis and Vázquez also conjectured that $z_0^2 \omega_{3+\epsilon}^{\frac{2}{3+\epsilon}} |\Omega|^{-\frac{2}{3+\epsilon}} \int_\Omega \sum |f_j|^2 dx$ is just the first term of an infinite series of extra terms that can be added to the right hand side of (1.2). This problem investigated by many authors, see for e.g. [1,5, 10,11,18,19,21,23, 24]. We also refer to [3, 26, 30, 31, 36, 40] which are excellent monographs on the topic. Note that in an attempt to improve, extend and unify several results in this direction, Ghoussoub and Moradifam [25] introduced the notion of a Bessel pair and studied its connections to Hardy inequalities. One of their results can be read as follows:

Theorem A. Let $0 \leq \epsilon \leq \infty, B_{1+\epsilon} = B(0,1+\epsilon)$ be a ball centered at the origin with radius $(1+\epsilon), V_j$ and W_j be positive $(1+\epsilon)^1$-functions on $(0,1+\epsilon)$ such that $\int_0^{1+\epsilon} \sum \frac{1}{r^{2+\epsilon} V_j(r)} dr = \infty$ and $\int_0^{1+\epsilon} \sum r^{2+\epsilon} V_j(r) dr < \infty$. Then

(1) If $\left(r^{2+\epsilon} V_j, r^{2+\epsilon} W_j\right)$ is a Bessel pair on $(0,1+\epsilon)$, then for all $f_j \in (1+\epsilon)_0^\infty(B_{1+\epsilon})$:

$$\int_{B_{1+\epsilon}} \sum V_j(|x|)|\nabla f_j|^2 dx \geq \int_{B_{1+\epsilon}} \sum W_j(|x|)|f_j|^2 dx. \qquad (1.3)$$

(2) If (1.3) holds for all $f_j \in (1+\epsilon)_0^\infty(B_{1+\epsilon})$, then $\left(r^{2+\epsilon} V_j, r^{2+\epsilon} c W_j\right)$ is a Bessel pair on $(0,1+\epsilon)$ for some $\epsilon \geq 0$.

Here we say that a couple of $(1+\epsilon)^1$-functions (V_j, W_j) is a Bessel pair on $(0,1+\epsilon)$ for some $-1 < \epsilon \leq \infty$ if the ordinary differential equation

$$\left(V_j y'\right)' + W_j y = 0 \qquad (1.4)$$

has a positive solution φ on the interval $(0,1+\epsilon)$.

Muckenhought pairs [37] have been used to study the necessary and sufficient conditions for the validity of the Hardy inequality on one dimensional space.

Hardy type inequalities have also been generalized to the cases with general distance functions, see [4, 10, 32, 33], in multipolar setting, see [9, 14, 15]. Now the following Hardy inequality has been first established on Riemannian manifold $(\mathbb{M}, g)$ by Carron [13]:

$$\int_{\mathbb{M}} \sum \rho^\alpha(x) |\nabla_g f_j|_g^2 d(V_j)_g \geq \left(\frac{\epsilon+\alpha}{2}\right)^2 \int_{\mathbb{M}} \sum \rho^\alpha(x) \frac{|f_j|^2}{\rho^2(x)} d(V_j)_g \qquad (1.5)$$

where $\alpha \in \mathbb{R}, (\epsilon+\alpha) > 0, f_j \in C_0^\infty(\mathbb{M} \setminus \rho^{-1}\{0\})$ and the weighted function ρ satisfies the eikonal equation $\left|\nabla_g \rho\right|_g = 1$ and $\Delta_g \rho \geq \frac{1+\epsilon}{\rho}$ for some $\epsilon \geq 0$. Here dV_g, ∇_g, Δ_g and $|\cdot|_g$ denote the volume element, gradient, Laplace-Beltrami operator and the length of a vector field with respect to the Riemannian metric g on $\mathbb{M}$, respectively. For developments see [6, 7, 17, 29].

When $\mathbb{M}$ is a $(3+\epsilon)$-dimensional Cartan-Hadamard manifold and $\rho = d(x, O)$ is the geodesic distance, then ρ satisfies all the aforementioned conditions. It was showed in [13] that

$$\int_{\mathbb{M}} \sum |\nabla_g f_j|_g^2 d(V_j)_g \geq \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{M}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_g . \quad (1.6)$$

The constant $\left(\frac{(3+\epsilon)-2}{2}\right)^2$ was verified to be optimal in [43]. When $\mathbb{M}$ is the hyperbolic space $\mathbb{H}^{3+\epsilon}$, we have

$$\int_{\mathbb{H}^{3+\epsilon}} \sum |\nabla_{\mathbb{H}} f_j|^2 d(V_j)_{\mathbb{H}} \geq \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_{\mathbb{H}} \quad (1.7)$$

where $\rho(x)$ is the geodesic distance on $\mathbb{H}^{3+\epsilon}$. On the other hand, it is well-known that on $\mathbb{H}^{3+\epsilon}$, the L^2-spectrum is $\left[\left(\frac{2+\epsilon}{2}\right)^2, \infty\right)$. Then, we have the Poincaré-Sobolev inequality

$$\int_{\mathbb{H}^{3+\epsilon}} \sum |\nabla_{\mathbb{H}} f_j|^2 d(V_j)_{\mathbb{H}} \geq \left(\frac{(2+\epsilon)-1}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum |f_j|^2 d(V_j)_{\mathbb{H}} \quad (1.8)$$

where $\left(\frac{2+\epsilon}{2}\right)^2$ is sharp and is never attained by nontrivial functions in $W_j^{1,2}(\mathbb{H}^{3+\epsilon})$. [2] investigated the finiteness and infiniteness of the discrete spectrum of the Schrödinger operator $-\Delta_{\mathbb{H}} + V_j$ and set up the following sharp improvements of the Poincaré-Sobolev inequality (1.8):

$$\int_{\mathbb{H}^{3+\epsilon}} \sum |\nabla_{\mathbb{H}} f_j|^2 d(V_j)_{\mathbb{H}} - \left(\frac{2+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum |f_j|^2 d(V_j)_{\mathbb{H}}$$
$$\geq \frac{1}{4} \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_{\mathbb{H}} + \frac{(2+\epsilon)(\epsilon)}{4} \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\sinh^2 \rho(x)} d(V_j)_{\mathbb{H}} . \quad (1.9)$$

Moreover, the operator $-\Delta_{\mathbb{H}} - \left(\frac{2+\epsilon}{2}\right)^2 - \frac{1}{4}\frac{1}{\rho^2(x)} - \frac{(2+\epsilon)(\epsilon)}{4}\frac{1}{\sinh^2 \rho(x)}$ is critical in $\mathbb{H}^{3+\epsilon} \setminus \{0\}$ in the sense that for any $W_j > \frac{1}{4r^2} + \frac{(2+\epsilon)(\epsilon)}{4}\frac{1}{\sinh^2 r}$, the inequality

$$\int_{\mathbb{H}^{3+\epsilon}} \sum |\nabla_{\mathbb{H}} f_j|^2 d(V_j)_{\mathbb{H}} - \left(\frac{2+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum |f_j|^2 d(V_j)_{\mathbb{H}}$$
$$\geq \int_{\mathbb{H}^{3+\epsilon}} \sum W_j |f_j|^2 d(V_j)_{\mathbb{H}} \quad \forall f_j \in C_0^\infty(\mathbb{H}^{3+\epsilon} \setminus \{0\})$$

is not valid. This Hardy-Poincaré-Sobolev inequality has also been studied on larger classes of manifolds in [7]. Recently, there has been progress of establishing higher order Hardy-Sobolev-Maz'ya inequalities on hyperbolic spaces using Fourier analysis on hyperbolic spaces (see Lu and Yang [34, 35]). It is also worth mentioning that the problems of improving Hardy type inequalities as well as other functional and geometric inequalities using the effect of curvature have been studied intensively recently. See [8,16,20,27,28,38,39,43].

We study the general Hardy type inequalities on Cartan-Hadamard manifolds. We will set up some general Hardy identities that can be used to derive several substantial improvements of the Hardy inequality on Cartan-Hadamard manifolds. Our equalities not only provide straightforward understandings of several Hardy type inequalities, but also explain the existence and nonexistence of nontrivial optimizers (see [43]).

Let $(\mathbb{M}, g)$ be a complete Riemannian manifold of dimension $(3+\epsilon)$. In a local coordinate system $\{x^i\}_{i=1}^{3+\epsilon}$, we can write

$$g = \sum g_{ij} dx^i dx^j.$$

The Laplace-Beltrami operator Δ_g with respect to the metric g may then be written as

$$\Delta_g = \sum \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^i}\left(\sqrt{\det(g_{ij})}\, g^{ij} \frac{\partial}{\partial x^j}\right)$$

where $(g^{ij}) = (g_{ij})^{-1}$. Denote by ∇_g the corresponding gradient. Then

$$\sum \langle \nabla_g f_j, \nabla_g h_j \rangle_g = \sum g^{ij} \frac{\partial f_j}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

We also denote

$$|\nabla_g f_j|_g = \sum \sqrt{\langle \nabla_g f_j, \nabla_g f_j \rangle_g}.$$

Fix a point $O \in \mathbb{M}$ and denote by $\rho(x) = d(x, O)$ for all $x \in \mathbb{M}$, where d denotes the geodesic distance on $\mathbb{M}$. Then $\rho(x)$ is Lipschitz continuous in $\mathbb{M}$.

For each point $O \in \mathbb{M}$, consider the exponential map $\exp_O : T_O\mathbb{M} \to \mathbb{M}$. For $X \in T_O\mathbb{M}$, let $\gamma(t)$ be the unique geodesic such that $\gamma(0) = O$ and $\gamma'(0) = X$. Then $\exp_O(tX) = \gamma(t)$ for $t > 0$. For small t, γ is the unique minimal geodesic joining the points O and $\exp_O(tX)$.

One can write

$$\mathbb{M} = \exp_O(U_O) \cup \mathrm{Cut}(O),$$

where $\mathrm{Cut}(O)$ denotes the cut locus of the point O and U_O is an open neighborhood of O in $T_O\mathbb{M}$. Furthermore, $\exp_O : U_O \to \exp_O(U_O)$ is a diffeomorphism and $(O) = \exp_O \partial U_O$. Also, the cut locus $Cut(O)$ has measure zero .

The distance function $\rho(x)$ is smooth on $\mathbb{M} \setminus (Cut(O) \cup \{O\})$ and it satisfies $\left|\nabla_g \rho(x)\right|_g = 1$ on $\mathbb{M} \setminus (\mathrm{Cut}(O) \cup \{O\})$.

For a Cartan-Hadamard manifold $(\mathbb{M}, g)$, the exponential map $\exp_O: T_O\mathbb{M} \to \mathbb{M}$ is a diffeomorphism and $\mathrm{Cut}(O) = \emptyset$, and then $\rho(x)$ is smooth in $\mathbb{M} \setminus \{O\}$ and $\left|\nabla_g \rho(x)\right|_g = 1$.

For any $(1+\epsilon) > 0$, denote by $B_{1+\epsilon}(O) = \{x \in \mathbb{M}: \rho(x) < \delta\}$ the geodesic ball in $\mathbb{M}$ with center at O and radius $(1+\epsilon)$. Now, we choose an orthonormal basis $\{u, e_2, \dots, e_{3+\epsilon}\}$ in $T_O\mathbb{M}$ and let $c(t) = \exp_O(tu)$ be a geodesic curve. Consider the Jacobi fields $\{Y_2(t), \dots, Y_{3+\epsilon}(t)\}$ satisfying $Y_i(0) = 0$ and $Y_i'(0) = e_i$, so that the volume density function written in geodesic polar coordinates can be given by

$$J(u,t) = t^{-(2+\epsilon)}\sqrt{\det\left(\left\langle Y_i(t), Y_j(t)\right\rangle\right)}, t > 0$$

We note that $J(u,t) \in C^\infty(T_O\mathbb{M} \setminus \{O\})$ and does not depend on $\{e_2, \dots, e_{3+\epsilon}\}$. By the definition of the density function $J(u,t)$, we have the polar coordinates on $\mathbb{M}$:

$$\int_{\mathbb{M}} \sum f_j(x) d\left(V_j\right)_g = \int_{\mathbb{S}^{2+\epsilon}} \int_0^\infty \sum f_j(\exp_O(tu)) J(u,t) t^{2+\epsilon} dt du$$

Here du denotes the canonical measure of the unit sphere of $T_O\mathbb{M}$.

For any functions f_j on $\mathbb{M}$, we also define the radial derivation $\partial_\rho = \frac{\partial}{\partial \rho}$ along the geodesic curve starting from O by

$$\partial_\rho f_j(x) = \sum \frac{d\left(f_j \circ \exp_O\right)}{dr}(\exp_O^{-1}(x)).$$

Here we denote $\frac{d}{dr}$ the radial derivation on $T_O\mathbb{M}$:

$$\frac{d}{dr} F_j(u) = \sum \left\langle \frac{u}{|u|}, \nabla F_j(u) \right\rangle.$$

We note that by Gauss's lemma, we have that $\left|\sum \partial_\rho f_j\right| \le \sum \left|\nabla_g f_j\right|_g$ for $f_j \in C^1(\mathbb{M} \setminus Cut(O))$.

We have the following Hardy type identities on the general complete Riemannian manifold $(\mathbb{M}, g)$:

Theorem 1.1 [43]. Let $(\mathbb{M}, g)$ be a complete Riemannian manifold of dimension $(3+\epsilon)$. Let $O \in \mathbb{M}$ and take $0 < (1+\epsilon) \le d(O, \mathrm{Cut}(O))$. Let V_j and W_j be positive C^1-functions on $(0, 1+\epsilon)$ such that $\left(r^{2+\epsilon} V_j, r^{2+\epsilon} W_j\right)$ is a Bessel pair on $(0, 1+\epsilon)$. Then we have the following identities for all $f_j \in C_0^\infty\left(\mathbb{M} \setminus (Cut(O) \cup \rho^{-1}\{0\})\right)$:

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\left|\nabla_g f_j\right|_g^2 d(V_j)_g - \int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f_j|^2 d(V_j)_g$$
$$= \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|\varphi^2(\rho(x))| \left|\nabla_g\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|_g^2 d(V_j)_g$$
$$- \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|f_j|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))}\frac{J'(u,\rho(x))}{J(u,\rho(x))} d(V_j)_g$$

and

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\left|\partial_\rho f_j\right|^2 d(V_j)_g - \int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f_j|^2 d(V_j)_g$$
$$= \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\varphi^2(\rho(x)) \left|\partial_\rho\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|^2 d(V_j)_g$$
$$- \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|f_j|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))}\frac{J'(u,\rho(x))}{J(u,\rho(x))} d(V_j)_g$$

Here $J'(u,t) = \frac{\partial J(u,t)}{\partial t}, x = \exp_O(\rho u)$ and φ is the positive solution of

$$\sum\left(r^{2+\epsilon}V_j(r)\varphi'(r)\right)' + \sum r^{2+\epsilon}W_j(r)\varphi(r) = 0.$$

We consider φ which have a zero at some $r = 1+\epsilon$, but are positive elsewhere and which satisfy the Bessel pair ODE on $(0,1+\epsilon) \cup (1+\epsilon,\infty)$. As the following theorem demonstrates, considering such φ allows one to establish global Hardy identities provided f_j is replaced by $f_j - f_j(\exp)$ and provided f_j satisfies (1.10). We note that on finite interval $(0,1+\epsilon)$, this function φ satisfies the condition in Theorem 1.1. However, on the infinite interval $(0,\infty)$, this φ is allowed to be degenerate or singular at $(1+\epsilon)$. We have [43]

Theorem 1.2. Let $(\mathbb{M},g)$ be a complete Riemannian manifold of dimension $(3+\epsilon)$. Let $O \in \mathbb{M}$ and take $0 < 1+\epsilon \le d(O, \text{Cut }(O))$ Assume that V_j and W_j are positive $(1+\epsilon)^1$-functions on $(0,1+\epsilon) \cup (1+\epsilon,\infty)$ such that the ordinary differential equation

$$\sum\left(V_j(r)r^{2+\epsilon}\varphi'(r)\right)' + \sum W_j(r)r^{2+\epsilon}\varphi(r) = 0$$

has a positive solution φ on $(0,1+\epsilon) \cup (1+\epsilon,\infty)$.

Then for all $f_j \in C_0^\infty\left(\mathbb{M} \setminus (Cut(O) \cup \rho^{-1}\{0\})\right)$ satisfying that for all $\in \mathbb{S}^{2+\epsilon}$:

$$\lim_{r\to 1+\epsilon} V_j \sum(r)\frac{\varphi'(r)}{\varphi(r)}\left|f_j(\exp_O(ru)) - f_j(\exp_O((1+\epsilon)u))\right|^2 = 0, \qquad (1.10)$$

we have

$$\int_{\mathbb{M}} \sum V_j(\rho(x)) \left|\nabla_g \left(f_j - f_j(\exp_O((1+\epsilon)u))\right)\right|_g^2 dx$$
$$- \int_{\mathbb{M}} \sum W_j(\rho(x)) |f_j - f_j(\exp_O((1+\epsilon)u))|^2 d(V_j)_g$$
$$= \int_{\mathbb{M}} \sum V_j(\rho(x)) \varphi^2(\rho(x)) \left|\nabla_g \left(\frac{f_j - f_j(\exp_O((1+\epsilon)u))}{\varphi(\rho(x))}\right)\right|_g^2 d(V_j)_g$$
$$- \int_{\mathbb{M}} \sum V_j(\rho(x)) |f_j - f_j(\exp_O((1+\epsilon)u))|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} \frac{J'(u,\rho)}{J(u,\rho)} d(V_j)_g$$

and

$$\int_{\mathbb{M}} \sum V_j(\rho(x)) \left|\partial_\rho \left(f_j - f_j(\exp_O((1+\epsilon)u))\right)\right|_g^2 dx$$
$$- \int_{\mathbb{M}} \sum W_j(\rho(x)) |f_j - f_j(\exp_O((1+\epsilon)u))|^2 d(V_j)_g$$
$$= \int_{\mathbb{M}} \sum V_j(\rho(x)) \varphi^2(\rho(x)) \left|\partial_\rho \left(\frac{f_j - f_j(\exp_O((1+\epsilon)u))}{\varphi(\rho(x))}\right)\right|^2 d(V_j)_g$$
$$- \int_{\mathbb{M}} \sum V_j(\rho(x)) |f_j - f_j(\exp_O((1+\epsilon)u))|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} \frac{J'(u,\rho)}{J(u,\rho)} d(V_j)_g$$

Here $J'(u,t) = \frac{\partial J(u,t)}{\partial t}$ and $x = \exp_O(\rho u)$.

By applying our main results to some explicit Bessel pairs on Cartan-Hadamard manifold, we obtain many interesting Hardy identities and inequalities. On the hyperbolic space, we obtain the following identities and inequalities that substantially improve (1.7) as consequences of our main results see [43]:

Theorem 1.3. For $f_j \in C_0^\infty(\mathbb{H}^{3+\epsilon})$:

$$\int_{\mathbb{H}^{3+\epsilon}} \sum |\nabla_{\mathbb{H}} f_j|^2 d(V_j)_{\mathbb{H}} - \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_{\mathbb{H}}$$
$$= \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left|\nabla_{\mathbb{H}} \left(\rho^{\frac{1+\epsilon}{2}}(x) f_j\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$+ \frac{(1+\epsilon)(2+\epsilon)}{2} \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{\rho(x)\cosh\rho(x) - \sinh\rho(x)}{\rho^2(x)\sinh\rho(x)} |f_j|^2 d(V_j)_{\mathbb{H}}$$

and

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}} - \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_{\mathbb{H}}$$
$$= \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left|\partial_\rho \left(\rho^{\frac{1+\epsilon}{2}}(x) f_j\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$+ \frac{(1+\epsilon)(2+\epsilon)}{2} \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{\rho(x)\cosh\rho(x) - \sinh\rho(x)}{\rho^2(x)\sinh\rho(x)} |f_j|^2 d(V_j)_{\mathbb{H}}$$

Obviously, our theorem gives the exact remainder and therefore provides the direct understanding for the Hardy inequality (1.7). Also, as a consequence of the above identities, we get that

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\nabla_{\mathbb{H}} f_j\right|^2 d(V_j)_{\mathbb{H}} \geq \int_{\mathbb{H}^{3+\epsilon}} \sum \left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}}$$
$$\geq \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_{\mathbb{H}}$$
$$+ \frac{(1+\epsilon)(2+\epsilon)}{2} \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{\rho(x)\cosh\rho(x) - \sinh\rho(x)}{\rho^2(x)\sinh\rho(x)} |f_j|^2 d(V_j)_{\mathbb{H}}.$$

Therefore, the operator $-\Delta_{\mathbb{H}} - \left(\frac{1+\epsilon}{2}\right)^2 \frac{1}{\rho^2(x)}$ is subcritical in $\mathbb{H}^{3+\epsilon} \setminus \{0\}$, which is in constrast to the situation in the Euclidean setting [26, Corollary 2.3.4].

We also present the exact remainder for the Hardy-Poincaré-Sobolev inequality (1.9) and thus sharpen the inequality (1.8) and illustrate more precise understanding of (1.9), see [43]:

Theorem 1.4. For $f_j \in C_0^\infty(\mathbb{H}^{3+\epsilon})$:

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\nabla_{\mathbb{H}} f_j\right|^2 d(V_j)_{\mathbb{H}}$$
$$- \int_{\mathbb{H}^{3+\epsilon}} \sum \left[\frac{(2+\epsilon)^2}{4} + \frac{1}{4}\frac{1}{\rho^2(x)} + \frac{(2+\epsilon)(\epsilon)}{4}\frac{1}{\sinh^2\rho(x)}\right] |f_j|^2 d(V_j)_{\mathbb{H}}$$
$$= \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{\rho(x)}{\sinh^{2+\epsilon}\rho(x)} \left|\nabla_{\mathbb{H}} \left(\frac{\sinh^{\frac{2+\epsilon}{2}}\rho(x) f_j}{\rho^{\frac{1}{2}}(x)}\right)\right|^2 d(V_j)_{\mathbb{H}},$$

and

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}} -$$

$$\int_{\mathbb{H}^{3+\epsilon}}\sum\left[\frac{(2+\epsilon)^2}{4}+\frac{1}{4}\frac{1}{\rho^2(x)}+\frac{(2+\epsilon)(\epsilon)}{4}\frac{1}{\sinh^2\rho(x)}\right]|f_j|^2 d(V_j)_{\mathbb{H}}$$
$$=\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\rho(x)}{\sinh^{2+\epsilon}\rho(x)}\left|\partial_\rho\left(\frac{\sinh^{\frac{2+\epsilon}{2}}\rho(x)f_j}{\rho^{\frac{1}{2}}(x)}\right)\right|^2 d(V_j)_{\mathbb{H}}$$

and

$$\int_{\mathbb{H}^{3+\epsilon}}\sum\left|\nabla_{\mathbb{H}}f_j\right|^2 d(V_j)_{\mathbb{H}}$$
$$\geq\int_{\mathbb{H}^{3+\epsilon}}\sum\left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}}$$
$$\geq\int_{\mathbb{H}^{3+\epsilon}}\sum\left[\frac{(2+\epsilon)^2}{4}+\frac{1}{4}\frac{1}{\rho^2(x)}+\frac{(2+\epsilon)(\epsilon)}{4}\frac{1}{\sinh^2\rho(x)}\right]|f_j|^2 d(V_j)_{\mathbb{H}}.$$

We also obtain the following Hardy inequalities on hyperbolic spaces in the spirit of Brezis-Vázquez [12], (see [43]):

Theorem 1.5. Let $0\leq\alpha\leq\frac{1+\epsilon}{2}$. For $f_j\in C_0^\infty(\mathbb{H}^{3+\epsilon})$:

$$\int_{0<\rho(x)<1+\epsilon}\sum\left|\nabla_{\mathbb{H}}f_j\right|^2 d(V_j)_{\mathbb{H}}-\left(\frac{(1+\epsilon)^2}{4}-\alpha^2\right)\int_{0<\rho(x)<1+\epsilon}\sum\frac{|f_j|^2}{\rho^2(x)}d(V_j)_{\mathbb{H}}$$
$$=\frac{z_\alpha^2}{1+\epsilon^2}\int_{0<\rho(x)<1+\epsilon}\sum|f_j|^2 d(V_j)_{\mathbb{H}}$$
$$+\int_{0<\rho(x)<1+\epsilon}\sum\frac{J_\alpha^2\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}{\rho^{1+\epsilon}(x)}\left|\nabla_{\mathbb{H}}\left(\frac{f_j}{\rho^{\frac{-(1+\epsilon)}{2}}(x)J_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$-\int_{0<\rho(x)<1+\epsilon}\sum\left(\frac{-(1+\epsilon)}{2\rho(x)}+\frac{z_\alpha}{1+\epsilon}\frac{J'_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}{J_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}\right)$$
$$\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

and

$$\int_{0<\rho(x)<1+\epsilon}\sum\left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}}-\left(\frac{(1+\epsilon)^2}{4}-\alpha^2\right)\int_{0<\rho(x)<1+\epsilon}\sum\frac{|f_j|^2}{\rho^2(x)}d(V_j)_{\mathbb{H}}$$
$$=\frac{z_\alpha^2}{(1+\epsilon)^2}\int_{0<\rho(x)<1+\epsilon}\sum|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$+\int_{0<\rho(x)<1+\epsilon}\sum\frac{J_\alpha^2\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}{\rho^{1+\epsilon}(x)}\left|\partial_\rho\left(\frac{f_j}{\rho^{\frac{-(1+\epsilon)}{2}}(x)J_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}\right)\right|^2 d(V_j)_{\mathbb{H}}$$

$$-((3+\epsilon)-1)\int_{0<\rho(x)<1+\epsilon}\sum\left(\frac{-(1+\epsilon)}{2\rho(x)}+\frac{z_\alpha}{1+\epsilon}\frac{J'_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}{J_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}\right)$$

$$\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

As a consequence of these identities, we get that

$$\int_{0<\rho(x)<1+\epsilon}\sum\left|\nabla_{\mathbb{H}}f_j\right|^2 d(V_j)_{\mathbb{H}} \geq \int_{0<\rho(x)<1+\epsilon}\sum\left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}}$$

$$\geq\left(\frac{(1+\epsilon)^2}{4}-\alpha^2\right)\int_{0<\rho(x)<1+\epsilon}\sum\frac{|f_j|^2}{\rho^2(x)}d(V_j)_{\mathbb{H}}+\frac{z_\alpha^2}{(1+\epsilon)^2}\int_{0<\rho(x)<1+\epsilon}\sum|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$-(2+\epsilon)\int_{0<\rho(x)<1+\epsilon}\sum\left(\frac{-(1+\epsilon)}{2\rho(x)}+\frac{z_\alpha}{1+\epsilon}\frac{J'_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}{J_\alpha\left(\frac{z_\alpha}{1+\epsilon}\rho(x)\right)}\right)$$

$$\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$\geq\left(\frac{((1+\epsilon)^2}{4}-\alpha^2\int_{0<\rho(x)<1+\epsilon}\sum\frac{|f_j|^2}{\rho^2(x)}d(V_j)_{\mathbb{H}}+\frac{z_\alpha^2}{(1+\epsilon)^2}\int_{0<\rho(x)<1+\epsilon}\sum|f_j|^2 d(V_j)_{\mathbb{H}}.\right.$$

Here z_α is the first zero of the Bessel function of the first kind $J_\alpha(z)$.

We use the main results on the Hardy identities to obtain several Hardy type inequalities and their improvements on Cartan-Hadamard manifolds. We will focus on deriving the Hardy identities and inequalities on hyperbolic spaces. We also provide the proofs of Theorem 1.3, Theorem 1.4 and Theorem 1.5. Proofs of main results (Theorem 1.1 and Theorem 1.2) will be presented latter.

2. Hardy inequalities on Cartan-Hadamard manifolds

We note that if the sectional curvature $K_{\mathbb{M}}=-b$, then $J(u,t)=J_b(t)$ does not depend on u. Moreover

$$J_b(t)=\begin{cases}1 & \text{if } b=0\\ \left(\frac{\sinh(\sqrt{b}t)}{\sqrt{b}t}\right)^{2+\epsilon} & \text{if } b>0\end{cases}.$$

Also, if $K_{\mathbb{M}}\leq -b\leq 0$, then by the Bishop-Gromov-Günther comparison theorem [22, page 172], we have that

$$\frac{J'(u,t)}{J(u,t)} \geq \frac{J_b'(t)}{J_b(t)} = \frac{2+\epsilon}{t}\mathbf{D}_b(t)$$

where $J'(u,t) = \frac{\partial J(u,t)}{\partial t}$,

$$\mathbf{D}_b(t) = \begin{cases} 0 & \text{if } t = 0 \\ t(1+\epsilon)\mathbf{t}_b(t) - 1 & \text{if } t > 0 \end{cases}$$

and

$$(1+\epsilon)\mathbf{t}_b(t) = \begin{cases} \frac{1}{t} & \text{if } b = 0 \\ \sqrt{b}\coth(\sqrt{b}t) & \text{if } b > 0 \end{cases}.$$

Therefore, we obtain the following Hardy type inequality as a direct consequence of Theorem 1.1 (see [43]):

Theorem 2.1. Let $(\mathbb{M}, g)$ be a Cartan-Hadamard manifold of dimension $(3+\epsilon)$ and let $O \in \mathbb{M}$. Let $0 < 1+\epsilon \leq \infty, V_j$ and W_j be positive $(1+\epsilon)^1 -$ functions on $(0,1+\epsilon)$ such that $(r^{2+\epsilon}V_j, r^{2+\epsilon}W_j)$ is a Bessel pair on $(0,1+\epsilon)$ with nonincreasing positive solution φ. Then for $f_j \in C_0^\infty(B_{1+\epsilon}(O) \setminus \rho^{-1}\{0\})$:

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\left|\nabla_g f_j\right|_g^2 d(V_j)_g - \int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f_j|^2 d(V_j)_g$$

$$\geq \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|\varphi^2(\rho(x))| \left|\nabla_g\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|_g^2 d(V_j)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|f|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))}\frac{J_b'(\rho(x))}{J_b(\rho(x))} d(V_j)_g$$

$$\geq \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|\varphi^2(\rho(x))| \left|\nabla_g\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|_g^2 d(V_j)_g$$

and

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\left|\partial_\rho f_j\right|^2 d(V_j)_g - \int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f_j|^2 d(V_j)_g$$

$$\geq \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\varphi^2(\rho(x)) \left|\partial_\rho\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|^2 d(V_j)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|f_j|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))}\frac{J_b'(\rho(x))}{J_b(\rho(x))} d(V_j)_g$$

$$\geq \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\varphi^2(\rho(x)) \left|\partial_\rho\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|^2 d(V_j)_g.$$

Proof. By the Bishop-Gromov-Günther comparison theorem [22, page 172], we get $\frac{J'(u,t)}{J(u,t)} \geq 0$, where $J'(u,t) = \frac{\partial J(u,t)}{\partial t}$. Since φ is a nonincreasing function, $-\varphi'(\rho(x))\frac{J'(u,\rho)}{J(u,\rho)} \geq -\varphi'(\rho(x))\frac{J_b'(\rho)}{J_b(\rho)} \geq 0$. Hence, we can apply Theorem 1.1 to get the desired result.

By applying Theorem 2.1 to particular Bessel pairs, we obtain several Hardy type inequalities with remainder terms on $\mathbb{M}$. These results are listed as follows (see [43]).

Corollary 2.1. for $\lambda < (1+\epsilon)$ and $f_j \in C_0^\infty(\mathbb{M} \setminus \rho^{-1}\{0\})$:

$$\int_{\mathbb{M}} \sum \frac{\left|\nabla_g f_j\right|_g^2}{\rho^\lambda(x)} d(V_j)_g - \left(\frac{(1+\epsilon)-\lambda}{2}\right)^2 \int_{\mathbb{M}} \sum \frac{|f_j|^2}{\rho^{\lambda+2}(x)} d(V_j)_g$$
$$\geq \int_{\mathbb{M}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left|\nabla_g \left(\rho^{\frac{(1+\epsilon)-\lambda}{2}}(x) f_j\right)\right|_g^2 d(V_j)_g$$

and

$$\int_{\mathbb{M}} \sum \frac{\left|\partial_\rho f_j\right|^2}{\rho^\lambda(x)} d(V_j)_g - \left(\frac{(1+\epsilon)-\lambda}{2}\right)^2 \int \sum \frac{|f_j|^2}{\rho^{\lambda+2}(x)} d(V_j)_g$$
$$\geq \int_{\mathbb{M}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left|\partial_\rho \left(\rho^{\frac{(1+\epsilon)-\lambda}{2}}(x) f_j\right)\right|^2 d(V_j)_g. \quad (2.1)$$

Proof. We apply Theorem 2.1 to the Bessel pair $\left(r^{2+\epsilon} r^{-\lambda}, r^{2+\epsilon} r^{-\lambda} \frac{((1+\epsilon)-\lambda)^2}{4} \frac{1}{r^2}\right)$ on $(0,\infty)$ with $\varphi(r) = r^{\frac{-(1+\epsilon)+\lambda}{2}}$ to get the desired results.

In the critical case $\lambda = (1+\epsilon)$, we have

Corollary 2.2. Let $\epsilon > -1$. We have for $f_j \in C_0^\infty(B_{1+\epsilon}(O) \setminus \rho^{-1}\{0\})$:

$$\int_{0<\rho(x)<1+\epsilon} \sum \frac{\left|\nabla_g f_j\right|_g^2}{\rho^{1+\epsilon}(x)} d(V_j)_g - \frac{1}{4} \int_{0<\rho(x)<1+\epsilon} \sum \frac{|f_j(x)|^2}{\rho^{3+\epsilon}(x) \left|\ln \frac{\rho(x)}{1+\epsilon}\right|^2} d(V_j)_g$$

$$\geq \int_{0<\rho(x)<1+\epsilon} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left|\ln \frac{\rho(x)}{1+\epsilon}\right| \left|\nabla_g \left(\frac{f_j(x)}{\sqrt{\left|\ln \frac{|x|}{1+\epsilon}\right|}}\right)\right|_g^2 d(V_j)_g$$

and

$$\int_{0<\rho(x)<1+\epsilon}\sum\frac{\left|\partial_\rho f_j\right|^2}{\rho^{1+\epsilon}(x)}d(V_j)_g-\frac{1}{4}\int_{0<\rho(x)<1+\epsilon}\sum\frac{|f_j(x)|^2}{\rho^{3+\epsilon}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2}d(V_j)_g$$

$$\geq\int_{0<\rho(x)<1+\epsilon}\sum\frac{1}{\rho^{1+\epsilon}(x)}\left|\ln\frac{\rho(x)}{1+\epsilon}\right|\left|\partial_\rho\left(\frac{f_j(x)}{\sqrt{\left|\ln\frac{|x|}{1+\epsilon}\right|}}\right)\right|^2 d(V_j)_g.$$

Proof. We apply Theorem 2.1 to the Bessel pair $\left(r^{2+\epsilon}\frac{1}{r^{1+\epsilon}},r^{2+\epsilon}\frac{1}{4r^{3+\epsilon}\left|\ln\frac{r}{1+\epsilon}\right|^2}\right)$ with $\varphi=\sqrt{\left|\ln\frac{r}{1+\epsilon}\right|}$

Actually, we can get the following version of the critical Hardy type inequalities on the whole space $\mathbb{M}$ which is more general than Corollary 2.2 (see [43]):

Corollary 2.3. Let $(1+\epsilon)>0$. For any $f_j\in C_0^\infty(\mathbb{M}\setminus\rho^{-1}\{0\})$, we have

$$\int_{\mathbb{M}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\left|\nabla_g\left(f_j(x)-f_j(\exp_O((1+\epsilon)u))\right)\right|_g^2 d(V_j)_g$$

$$\geq\frac{1}{4}\int_{\mathbb{M}}\sum\frac{\left|f_j(x)-f_j(\exp_O((1+\epsilon)u))\right|^2}{\rho^{3+\epsilon}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2}d(V_j)_g$$

$$+\int_{\mathbb{M}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\left|\ln\frac{\rho(x)}{1+\epsilon}\right|\left|\nabla\left(\frac{f_j(x)-f_j(\exp_O((1+\epsilon)u))}{\sqrt{\left|\ln\frac{|x|}{1+\epsilon}\right|}}\right)\right|_g^2 d(V_j)_g$$

and

$$\int_{\mathbb{M}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\left|\partial_\rho\left(f_j(x)-f_j(\exp_O((1+\epsilon)u))\right)\right|^2 d(V_j)_g$$

$$\geq\frac{1}{4}\int_{\mathbb{M}}\sum\frac{\left|f_j(x)-f_j(\exp_O((1+\epsilon)u))\right|^2}{\rho^{3+\epsilon}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2}d(V_j)_g$$

$$+\int_{\mathbb{M}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\left|\ln\frac{\rho(x)}{1+\epsilon}\right|\left|\partial_\rho\left(\frac{f_j(x)-f_j(\exp_O((1+\epsilon)u))}{\sqrt{\left|\ln\frac{|x|}{1+\epsilon}\right|}}\right)\right|^2 d(V_j)_g.$$

Proof. Apply the Theorem 1.2 to $V_j(r) = \frac{1}{r^{1+\epsilon}}, W_j = \frac{1}{4r^{3+\epsilon}\left|\ln\frac{r}{1+\epsilon}\right|^2}$ and $\varphi = \sqrt{\left|\ln\frac{r}{1+\epsilon}\right|}$. Note that for any $f_j \in C_0^\infty(\mathbb{M} \setminus \rho^{-1}\{0\})$ and any $\in \mathbb{S}^{2+\epsilon}$:

$$\lim_{r\to 1+\epsilon}\sum\left|V_j(r)\frac{\varphi'(r)}{\varphi(r)}\right|\left|f_j(\exp_O(ru)) - f_j(\exp_O((1+\epsilon)u))\right|^2\Big|$$

$$= \lim_{r\to 1+\epsilon}\sum\frac{1}{r^{1+\epsilon}}\left|\frac{\left(\sqrt{\left|\ln\frac{r}{1+\epsilon}\right|}\right)'}{\sqrt{\left|\ln\frac{r}{1+\epsilon}\right|}}\right|\left|f_j(\exp_O(ru)) - f_j(\exp_O((1+\epsilon)u))\right|^2$$

$$\lesssim \lim_{r\to 1+\epsilon}\frac{1}{\left|\ln\frac{r}{1+\epsilon}\right|}((1+\epsilon)-r)^2 = 0.$$

Now, by combining Corollaries 2.1 and 2.3, we obtain (see [43])

Corollary 2.4. Let $(\mathbb{M}, g)$ be a Cartan-Hadamard manifold of dimension $3+\epsilon$. For $\lambda < (3+\epsilon) - 2$ and $f_j \in C_0^\infty(\mathbb{M} \setminus \rho^{-1}\{0\})$, we have

$$\int_{\mathbb{M}}\sum\frac{\left|\partial_\rho f_j\right|^2}{\rho^\lambda(x)}d(V_j)_g - \left(\frac{(1+\epsilon)-\lambda}{2}\right)^2\int_{\mathbb{M}}\sum\frac{|f_j|^2}{\rho^{\lambda+2}(x)}d(V_j)_g$$
$$\geq \frac{1}{4}\sup_{1+\epsilon>0}\int_{\mathbb{M}}\sum\frac{\left|f_j(x) - (1+\epsilon)^{\frac{(1+\epsilon)-\lambda}{2}}f_j(\exp_O((1+\epsilon)u))\rho^{-\frac{(1+\epsilon)-\lambda}{2}}(x)\right|^2}{\rho^{\lambda+2}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2}d(V_j)_g. \quad (2.2)$$

Proof. From Corollaries 2.1 and 2.3, we have

$$\int_{\mathbb{M}}\sum\frac{\left|\partial_\rho f_j\right|^2}{\rho^\lambda(x)}d(V_j)_g - \left(\frac{(1+\epsilon)-\lambda}{2}\right)^2\int_{\mathbb{M}}\sum\frac{|f_j|^2}{\rho^{\lambda+2}(x)}d(V_j)_g$$

$$\geq \int_{\mathbb{M}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left| \partial_\rho \left(\rho^{\frac{(1+\epsilon)-\lambda}{2}}(x) f_j \right) \right|^2 d(V_j)_g$$

$$= \int_{\mathbb{M}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left| \partial_\rho \left(\rho^{\frac{(1+\epsilon)-\lambda}{2}}(x) f_j - (1+\epsilon)^{\frac{(1+\epsilon)-\lambda}{2}} f_j(\exp_O((1+\epsilon)u)) \right) \right|^2 d(V_j)_g$$

$$\geq \frac{1}{4} \int_{\mathbb{M}} \sum \frac{\left| \rho^{\frac{(1+\epsilon)-\lambda}{2}}(x) f_j(x) - (1+\epsilon)^{\frac{(1+\epsilon)-\lambda}{2}} f_j(\exp_O((1+\epsilon)u)) \right|^2}{\rho^{3+\epsilon}(x) \left| \ln \frac{\rho(x)}{1+\epsilon} \right|^2} d(V_j)_g$$

$$+ \int_{\mathbb{M}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left| \ln \frac{\rho(x)}{1+\epsilon} \right|$$

$$\left| \partial_\rho \left(\frac{\rho^{\frac{(1+\epsilon)-\lambda}{2}} f_j(x) - (1+\epsilon)^{\frac{(1+\epsilon)-\lambda}{2}} f_j(\exp_O((1+\epsilon)u))}{\sqrt{\left| \ln \frac{|x|}{1+\epsilon} \right|}} \right) \right|^2 d(V_j)_g .$$

Note that the "virtual" optimizers of the weighted Hardy inequality (2.1) have the form $\psi(\exp_O(u))\rho^{-\frac{(1+\epsilon)-\lambda}{2}}(x)$ for some function $\psi: \mathbb{S}^{2+\epsilon} \to \mathbb{R}$. These optimizers are virtual in the sense that, if equality were to hold in the Hardy inequality

$$\int_{\mathbb{M}} \sum \frac{\left|\partial_\rho f_j\right|^2}{\rho^\lambda(x)} d(V_j)_g \geq \left(\frac{(1+\epsilon)-\lambda}{2} \right)^2 \int_{\mathbb{M}} \sum \frac{|f_j|^2}{\rho^{\lambda+2}(x)} d(V_j)_g$$

given by (2.1), then the remainder term would vanish, i.e., $\sum \partial_\rho \left(\rho^{\frac{(1+\epsilon)-\lambda}{2}}(x) f_j \right) = 0$. Therefore, (2.2) can be read as a stability version of the weighted Hardy inequality (2.1).

We also obtain the following Hardy inequality in the spirit of Brezis and Vázquez [12] (see [43]):

Corollary 2.5. Let $(\mathbb{M}, g)$ be a Cartan-Hadamard manifold of dimension $3+\epsilon$. For any $1+\epsilon > 0$ and $\lambda \leq 1+\epsilon$, we have for $f_j \in C_0^\infty(B_{1+\epsilon}(O) \setminus \rho^{-1}\{0\})$:

$$\int_{B_{1+\epsilon}(O)} \sum \frac{\left|\nabla_g f_j\right|_g^2}{\rho^\lambda(x)} d(V_j)_g - \left(\frac{(1+\epsilon)-\lambda}{2} \right)^2 \int_{B_{1+\epsilon}(O)} \sum \frac{|f_j|^2}{\rho^{\lambda+2}(x)} d(V_j)_g$$

$$\geq \frac{z_0^2}{(1+\epsilon)^2} \int_{B_{1+\epsilon}(O)} \sum \frac{|f_j|^2}{\rho^\lambda(x)} d(V_j)_g$$

$$+\int_{B_{1+\epsilon}(O)}\sum\left|\frac{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}{\rho(x)^{\frac{(1+\epsilon)-\lambda}{2}}}\right|^2\left|\nabla_g\left(\frac{\rho(x)^{\frac{(1+\epsilon)-\lambda}{2}}}{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}f_j\right)\right|_g^2 d(V_j)_g$$

and

$$\int_{B_{1+\epsilon}(O)}\sum\frac{|\partial_\rho f_j|^2}{\rho^\lambda(x)}d(V_j)_g-\left(\frac{(1+\epsilon)-\lambda}{2}\right)^2\int_{B_{1+\epsilon}(O)}\sum\frac{|f_j|^2}{\rho^{\lambda+2}(x)}d(V_j)_g$$
$$\geq\frac{z_0^2}{(1+\epsilon)^2}\int_{B_{1+\epsilon}(O)}\sum\frac{|f_j|^2}{\rho^\lambda(x)}d(V_j)_g$$
$$+\int_{B_{1+\epsilon}(O)}\sum\left|\frac{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}{\rho(x)^{\frac{(1+\epsilon)-\lambda}{2}}}\right|^2\left|\partial_\rho\left(\frac{\rho(x)^{\frac{(1+\epsilon)-\lambda}{2}}}{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}f_j\right)\right|^2 d(V_j)_g.$$

Proof. For any $1+\epsilon>0,\left(r^{2+\epsilon-\lambda},r^{2+\epsilon-\lambda}\left[\frac{((1+\epsilon)-\lambda)^2}{4}\frac{1}{r^2}+\frac{z_0^2}{(1+\epsilon)^2}\right]\right)$ is a Bessel pair on $(0,1+\epsilon)$ with $\varphi(r)=r^{-\frac{(1+\epsilon)-\lambda}{2}}J_0\left(\frac{rz_0}{1+\epsilon}\right)=r^{-\frac{(1+\epsilon)-\lambda}{2}}J_{0;1+\epsilon}(r)$. Here $z_0=2.4048\ldots$ is the first zero of the Bessel function $J_0(z)$. Note that $\varphi(r)$ is nonincreasing since $1+\epsilon-\lambda\geq 0$.

3. Hardy Inequalities on Hyperbolic Spaces

We will investigate the Hardy identities and inequalities on the hyperbolic space $\mathbb{H}^{3+\epsilon}$, which is the most important example of Cartan-Hadamard manifold. We use the Poincaré ball model of the hyperbolic space $\mathbb{H}^{3+\epsilon}$. That is, the unit ball in $\mathbb{R}^{3+\epsilon}$ centered at the origin and equipped with the metric

$$ds^2=\frac{4\sum dx_i^2}{(1-r^2)^2}.$$

Also

$$d(V_j)_{\mathbb{H}}=\frac{2^{3+\epsilon}}{(1-r^2)^{3+\epsilon}}dx,$$
$$\nabla_{\mathbb{H}}=\left(\frac{1-r^2}{2}\right)^2\nabla,$$

where ∇ denotes the Euclidean gradient. Therefore

$$\int_{\mathbb{H}^{3+\epsilon}}\sum|\nabla_{\mathbb{H}}u|^2 d(V_j)_{\mathbb{H}}=\int_B|\nabla u|^2\frac{2^{1+\epsilon}}{(1-|x|^2)^{1+\epsilon}}dx.$$

We also recall that the geodesic distance from x to 0 is $\rho(x)=\ln\frac{1+|x|}{1-|x|}$. That is $|x|=\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}$. By using the Poincaré ball model and applying our Theorem 1.1

for the unit ball on the Euclidean space $\mathbb{R}^{3+\epsilon}$, we get the following identity (see [43]):

Theorem 3.1. If $\sum\left(r^{2+\epsilon}\frac{1}{(1-r^2)^{1+\epsilon}}V_j, r^{2+\epsilon}\frac{4}{(1-r^2)^{3+\epsilon}}W_j\right)$ is a Bessel pair on $(0,1)$, then we have for $f_j \in C_0^\infty(\mathbb{H}^{3+\epsilon}\setminus\{0\})$:

$$\int_{\mathbb{H}^{3+\epsilon}}\sum V_j\left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right)\left|\nabla_{\mathbb{H}}f_j\right|^2 d(V_j)_{\mathbb{H}} - \int_{\mathbb{H}^{3+\epsilon}}\sum W_j\left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right)|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$= \int_{\mathbb{H}^{3+\epsilon}}\sum V_j\left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right)\left|\varphi^2\left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right)\right|\left|\nabla_{\mathbb{H}}\left(\frac{f_j}{\varphi\left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right)}\right)\right|^2 d(V_j)_{\mathbb{H}}$$

Here φ is the positive solution of

$$\sum\left(r^{2+\epsilon}\frac{1}{(1-r^2)^{1+\epsilon}}V_j(r)\varphi'(r)\right)' + \sum r^{2+\epsilon}\frac{4}{(1-r^2)^{3+\epsilon}}W_j(r)\varphi(r) = 0$$

on $(0,1)$.

Proof. Since $\sum\left(r^{2+\epsilon}\frac{1}{(1-r^2)^{1+\epsilon}}V_j, r^{3+\epsilon-1}\frac{4}{(1-r^2)^{3+\epsilon}}W_j\right)$ is a Bessel pair on $(0,1)$ with solution φ, we get

$$\int_{B(0,1)}\sum\frac{1}{(1-|x|^2)^{1+\epsilon}}V_j(|x|)|\nabla f_j|^2 dx - \int_{B(0,1)}\sum\frac{4}{(1-|x|^2)^{3+\epsilon}}W_j(|x|)|f_j|^2 dx$$

$$= \int_{B(0,1)}\sum\frac{1}{(1-|x|^2)^{1+\epsilon}}V_j(|x|)|\varphi^2(|x|)|\left|\nabla\left(\frac{f_j}{\varphi(|x|)}\right)\right|^2 dx.$$

Equivalently,

$$\int_{\mathbb{H}^{3+\epsilon}}\sum V_j(|x|)\left|\nabla_{\mathbb{H}}f_j\right|^2 d(V_j)_{\mathbb{H}} - \int_{\mathbb{H}^{3+\epsilon}}\sum W_j(|x|)|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$= \int_{\mathbb{H}^{3+\epsilon}}\sum V_j(|x|)|\varphi^2(|x|)|\left|\nabla_{\mathbb{H}}\left(\frac{f_j}{\varphi(|x|)}\right)\right|^2 d(V_j)_{\mathbb{H}}.$$

Now, let

$$\begin{aligned} F_j(r) &= \frac{(1-r^2)^{1+\epsilon}}{r^{2+\epsilon}} \\ G(r) &= \int_r^1 F_j(t)dt \\ (V_j)_2(r) &= \frac{F_j^2(r)(1-r^2)^2}{4((3+\epsilon)-2)^2 G^2(r)}. \end{aligned}$$

Then we have (see [43])

Corollary 3.1. For $f_j \in C_0^\infty(\mathbb{H}^{3+\epsilon})$:

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\nabla_{\mathbb{H}} f_j\right|^2 d(V_j)_{\mathbb{H}} - \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum (V_j)_2 \left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right) |f_j|^2 d(V_j)_{\mathbb{H}}$$

$$= \int_{\mathbb{H}^{3+\epsilon}} \sum \left|G\left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right)\right| \left|\nabla_{\mathbb{H}}\left(\frac{f_j}{\sqrt{G\left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right)}}\right)\right|^2 d(V_j)_{\mathbb{H}}.$$

As a consequence, we obtain the following Hardy type inequality that has been studied in [41]:

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\nabla_{\mathbb{H}} f_j\right|^2 d(V_j)_{\mathbb{H}} \geq \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum (V_j)_2 \left(\frac{e^{\rho(x)}-1}{e^{\rho(x)}+1}\right) |f_j|^2 d(V_j)_{\mathbb{H}}.$$

Proof. We note that $\frac{G}{(3+\epsilon)\omega_{2+\epsilon}}$ is a fundamental solution of the hyperbolic Laplacian. We have that $\sum\left(r^{2+\epsilon}\frac{1}{(1-r^2)^{1+\epsilon}}, r^{2+\epsilon}\frac{F_j^2(r)}{4G^2(r)}\frac{1}{(1-r^2)^{1+\epsilon}}\right)$ is a Bessel pair on $(0,1)$ with $\varphi = \sum\sqrt{G(r)}$.

That is $\left(r^{2+\epsilon}\frac{1}{(1-r^2)^{1+\epsilon}}\varphi'\right)' + \sum r^{2+\epsilon}\frac{F_j^2(r)}{4G^2(r)}\frac{1}{(1-r^2)^{1+\epsilon}}\varphi = 0$. Indeed, a direct computation shows

$$r^{2+\epsilon}\frac{1}{(1-r^2)^{1+\epsilon}}\varphi'$$

$$= r^{2+\epsilon}\sum\frac{1}{(1-r^2)^{1+\epsilon}}\frac{G'(r)}{2\sqrt{G(r)}} = -\frac{r^{2+\epsilon}}{(1-r^2)^{1+\epsilon}}\frac{(1-r^2)^{1+\epsilon}}{r^{2+\epsilon}}\sum\frac{1}{2\sqrt{G(r)}}$$

$$= -\sum\frac{1}{2\sqrt{G(r)}}$$

Thus

$$\left(r^{2+\epsilon}\frac{1}{(1-r^2)^{1+\epsilon}}\varphi'\right)' = -\sum\left(\frac{1}{2\sqrt{G(r)}}\right)'$$

$$= \frac{1}{4}\sum\frac{\sqrt{G(r)}G'(r)}{G^2(r)}$$

$$= -\frac{1}{4}\sum\frac{\sqrt{G(r)}}{G^2(r)}\frac{(1-r^2)^{1+\epsilon}}{r^{2+\epsilon}}$$

$$= -r^{2+\epsilon}\sum\frac{F_j^2(r)}{4G^2(r)}\frac{1}{(1-r^2)^{1+\epsilon}}\varphi$$

Hence by Theorem 3.1, we obtain

$$\int_{\mathbb{H}^{3+\epsilon}}\sum\left|\nabla_{\mathbb{H}}f_j\right|^2 d(V_j)_{\mathbb{H}}=\left(\frac{2+\epsilon}{2}\right)^2\int_{\mathbb{H}^{3+\epsilon}}\sum(V_j)_2|f_j|^2 d(V_j)_{\mathbb{H}}$$
$$+\int_{\mathbb{H}^{3+\epsilon}}\sum|G(|x|)|\left|\nabla_{\mathbb{H}}\left(\frac{f_j}{\sqrt{G(|x|)}}\right)\right|^2 d(V_j)_{\mathbb{H}}.$$

Now, we note that $K_{\mathbb{H}^{3+\epsilon}}=-1$. Therefore $J(u,t)=J_1(t)$ does not depend on u. Moreover

$$J_1(t)=\left(\frac{\sinh t}{t}\right)^{2+\epsilon}.$$

and

$$\frac{J'(u,t)}{J(u,t)}=\frac{J_1'(t)}{J_1(t)}=\frac{2+\epsilon}{t}(t\coth t-1).$$

Therefore, we can rewrite Theorem 1.1 as follows (see [43]):

Theorem 3.2. Let $\left(r^{2+\epsilon}V_j,r^{2+\epsilon}W_j\right)$ be a Bessel pair on $(0,1+\epsilon)$. Then we have the following identities

$$\int_{B_{1+\epsilon}}\sum V_j(\rho(x))\left|\nabla_{\mathbb{H}}f_j\right|^2 d(V_j)_{\mathbb{H}}-\int_{B_{1+\epsilon}}\sum W_j(\rho(x))|f_j|^2 d(V_j)_{\mathbb{H}}$$
$$=\int_{B_{1+\epsilon}}\sum V_j(\rho(x))|\varphi^2(\rho(x))|\left|\nabla_{\mathbb{H}}\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$-(2+\epsilon)\int_{B_{1+\epsilon}}\sum V_j(\rho(x))\frac{\varphi'(\rho(x))}{\varphi(\rho(x))}\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

and

$$\int_{B_{1+\epsilon}}\sum V_j(\rho(x))\left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}}-\int_{B_{1+\epsilon}}\sum W_j(\rho(x))|f_j|^2 d(V_j)_{\mathbb{H}}$$
$$=\int_{B_{1+\epsilon}}\sum V_j(\rho(x))|\varphi^2(\rho(x))|\left|\partial_\rho\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$-(2+\epsilon)\int_{B_{1+\epsilon}}\sum V_j(\rho(x))\frac{\varphi'(\rho(x))}{\varphi(\rho(x))}\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}.$$

By applying Theorem 3.2 to some explicit Bessel pairs, we obtain several improvements of the Hardy inequalities on hyperbolic spaces.

Proof of Theorem 1.3 [43]. We apply Theorem 3.2 to the Bessel pair $\left(r^{2+\epsilon},r^{2+\epsilon}\left(\frac{1+\epsilon}{2}\right)^2\frac{1}{r^2}\right)$ on $(0,\infty)$. Note that in this case $\varphi(r)=r^{-\frac{1+\epsilon}{2}}$.

From Theorem 1.3, we can deduce the Hardy-Poincaré-Sobolev identities and inequalities that provide improved versions with exact remainder terms of the Hardy-Poincaré Sobolev inequalities studied in [2,7].

Proof of Theorem 1.4 [43]. Let $\Psi(r) = \frac{r}{\sinh r}$ and $\Phi(r) = \Psi(r)^{\frac{2+\epsilon}{2}} = \left(\frac{r}{\sinh r}\right)^{\frac{2+\epsilon}{2}}$ and $W_j(r) = -\frac{(r\Phi'(r))'}{r^{2+\epsilon}\Phi}$. Then noting that

and

$$\Psi'(r) = \frac{(1 - r\coth r)}{\sinh r}$$

Hence

$$\begin{aligned}\Phi'(r) &= \frac{2+\epsilon}{2}\Psi(r)^{\frac{\epsilon}{2}}\Psi'(r)\\ &= \frac{2+\epsilon}{2}\left(\frac{r}{\sinh r}\right)^{\frac{\epsilon}{2}}\frac{(1 - r\coth r)}{\sinh r}\end{aligned}$$

and

$$\frac{\Phi'(r)}{\Phi(r)} = \frac{2+\epsilon}{2}\frac{1 - r\coth r}{r}$$

$$r^{1+\epsilon}W_j(r)$$

$$= -\frac{2+\epsilon}{2}\frac{\left(r\left(\frac{r}{\sinh r}\right)^{\frac{\epsilon}{2}}\frac{(1 - r\coth r)}{\sinh r}\right)'}{r\left(\frac{r}{\sinh r}\right)^{\frac{2+\epsilon}{2}}}$$

$$= -\frac{2+\epsilon}{2}\left[\frac{(1 - r\coth r)}{r^2} + \frac{\epsilon}{2}\frac{(1 - r\coth r)^2}{r^2} + \coth^2 r - 2\frac{\coth r}{r} + \frac{1}{\sinh^2 r}\right]$$

Note that $\left(r^{2+\epsilon}\frac{1}{r^{1+\epsilon}}\Phi'(r)\right)' + r^{2+\epsilon}W_j(r)\Phi(r) = 0$, we have by Theorem 3.2 that

$$\begin{aligned}\int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)}\left|\nabla_{\mathbb{H}}\left(\rho^{\frac{1+\epsilon}{2}}(x)f_j\right)\right|_{\mathbb{H}}^2 d(V_j)_{\mathbb{H}}\\ = \int_{\mathbb{H}^{3+\epsilon}} \sum W_j(\rho(x))\left|\rho^{\frac{1+\epsilon}{2}}(x)f_j\right|^2 d(V_j)_{\mathbb{H}}\\ + \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)}\Phi^2(\rho(x))\left|\nabla_{\mathbb{H}}\left(\frac{\rho^{\frac{1+\epsilon}{2}}(x)f_j}{\Phi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}\\ + \frac{(2+\epsilon)^2}{2}\int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)}\left(\frac{\rho(x)\coth\rho(x) - 1}{\rho(x)}\right)^2\left|\rho^{\frac{1+\epsilon}{2}}(x)f_j\right|^2 d(V_j)_{\mathbb{H}}.\end{aligned}$$

Therefore, from Theorem 1.3, we obtain

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\nabla_{\mathbb{H}} f_j\right|^2 d(V_j)_{\mathbb{H}} = \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_{\mathbb{H}}$$
$$+ \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \left|\nabla_{\mathbb{H}} \left(\rho^{\frac{1+\epsilon}{2}}(x) f_j\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$+ \frac{(1+\epsilon)(2+\epsilon)}{2} \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{\rho(x)\coth\rho(x) - 1}{\rho^2(x)} |f_j|^2 d(V_j)_{\mathbb{H}}$$
$$= \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \Phi^2(\rho(x)) \left|\nabla_{\mathbb{H}} \left(\frac{\rho^{\frac{1+\epsilon}{2}}(x) f_j}{\Phi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$+ \left(\frac{1+\epsilon}{2}\right)^2 \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{|f_j|^2}{\rho^2(x)} d(V_j)_{\mathbb{H}} + \frac{(1+\epsilon)(2+\epsilon)}{2} \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{\rho(x)\coth\rho(x) - 1}{\rho^2(x)} |f_j|^2 d(V_j)_{\mathbb{H}}$$
$$+ \frac{(2+\epsilon)^2}{2} \int_{\mathbb{H}^{3+\epsilon}} \sum \left(\frac{\rho(x)\coth\rho(x) - 1}{\rho(x)}\right)^2 |f_j|^2 + \int_{\mathbb{H}^{3+\epsilon}} \sum \rho^{1+\epsilon}(x) W(\rho(x)) |f_j|^2 d(V_j)_{\mathbb{H}}$$
$$= \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \Phi^2(\rho(x)) \left|\nabla_{\mathbb{H}} \left(\frac{\rho^{\frac{1+\epsilon}{2}}(x) f_j}{\Phi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$+ \int_{\mathbb{H}^{3+\epsilon}} \sum \left[\begin{array}{l} \left(\frac{1+\epsilon}{2}\right)^2 \frac{1}{\rho^2(x)} + \frac{(1+\epsilon)(2+\epsilon)}{2} \frac{\rho(x)\coth\rho(x) - 1}{\rho^2(x)} \\ - \frac{2+\epsilon}{2} \left[\frac{(1-\rho(x)\coth\rho(x))}{\rho^2(x)} + \frac{\epsilon}{2} \frac{(1-\rho(x)\coth\rho(x))^2}{\rho^2(x)} \right. \\ + \frac{(2+\epsilon)^2}{2} \left(\frac{\rho(x)\coth\rho(x) - 1}{\rho(x)}\right)^2 \\ \left. + \coth^2\rho(x) - 2\frac{\coth\rho(x)}{\rho(x)} + \frac{1}{\sinh^2 r}\right] \end{array}\right] |f_j|^2 d(V_j)_{\mathbb{H}}$$
$$= \int_{\mathbb{H}^{3+\epsilon}} \sum \frac{1}{\rho^{1+\epsilon}(x)} \Phi^2(\rho(x)) \left|\nabla_{\mathbb{H}} \left(\frac{\rho^{\frac{1+\epsilon}{2}}(x) f_j}{\Phi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}$$
$$+ \int_{\mathbb{H}^{3+\epsilon}} \sum \left[\frac{(2+\epsilon)^2}{4} + \frac{1}{4}\frac{1}{\rho^2(x)} + \frac{(2+\epsilon)(\epsilon)}{4\sinh^2\rho(x)}\right] |f_j|^2 d(V_j)_{\mathbb{H}}$$

In other words,

$$\int_{\mathbb{H}^{3+\epsilon}} \sum \left|\nabla_{\mathbb{H}} f_j\right|^2 d(V_j)_{\mathbb{H}}$$

$$-\int_{\mathbb{H}^{3+\epsilon}}\sum\left[\frac{(2+\epsilon)^2}{4}+\frac{1}{4}\frac{1}{\rho^2(x)}+\frac{(2+\epsilon)(\epsilon)}{4}\frac{1}{\sinh^2\rho(x)}\right]|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$=\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\Phi^2(\rho(x))\left|\nabla_{\mathbb{H}}\left(\frac{\rho^{\frac{1+\epsilon}{2}}(x)f_j}{\Phi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}.$$

Similarly, we also get

$$\int_{\mathbb{H}^{3+\epsilon}}\sum\left|\partial_\rho f_j\right|^2 d(V_j)_{\mathbb{H}}-\int_{\mathbb{H}^{3+\epsilon}}\sum\left[\frac{(2+\epsilon)^2}{4}+\frac{1}{4}\frac{1}{\rho^2(x)}+\frac{(2+\epsilon)(\epsilon)}{4}\frac{1}{\sinh^2\rho(x)}\right]|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$=\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\Phi^2(\rho(x))\left|\partial_\rho\left(\frac{\rho^{\frac{1+\epsilon}{2}}(x)f_j}{\Phi(\rho(x))}\right)\right|^2 d(V_j)_{\mathbb{H}}.$$

Corollary 3.2 [43]. We have

$$\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\left|\nabla_{\mathbb{H}}f_j\right|^2}{\rho^\lambda(x)}d(V_j)_{\mathbb{H}}-\frac{((1+\epsilon)-\lambda)^2}{4}\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{|f_j|^2}{\rho^{\lambda+2}(x)}d(V_j)_{\mathbb{H}}$$

$$=\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\left|\nabla_{\mathbb{H}}\left(\rho^{\frac{(1+\epsilon)-\lambda}{2}}(x)f_j\right)\right|^2 d(V_j)_{\mathbb{H}}$$

$$+\frac{((1+\epsilon)-\lambda)(2+\epsilon)}{2}\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho^{\lambda+2}(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

and

$$\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\left|\partial_\rho f_j\right|^2}{\rho^\lambda(x)}d(V_j)_{\mathbb{H}}-\frac{((1+\epsilon)-\lambda)^2}{4}\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{|f_j|^2}{\rho^{\lambda+2}(x)}d(V_j)_{\mathbb{H}}$$

$$=\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{1}{\rho^{1+\epsilon}(x)}\left|\partial_\rho\left(\rho^{\frac{(1+\epsilon)-\lambda}{2}}(x)f_j\right)\right|^2 d(V_j)_{\mathbb{H}}$$

$$+\frac{((1+\epsilon)-\lambda)(2+\epsilon)}{2}\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho^{\lambda+2}(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}.$$

As a consequence of these identities, we get that for $\lambda<(3+\epsilon)-2$:

$$\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\left|\nabla_{\mathbb{H}}f_j\right|^2}{\rho^\lambda(x)}d(V_j)_{\mathbb{H}}\geq\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\left|\partial_\rho f_j\right|^2}{\rho^\lambda(x)}d(V_j)_{\mathbb{H}}$$

$$\geq\frac{((1+\epsilon)-\lambda)^2}{4}\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{|f_j|^2}{\rho^{\lambda+2}(x)}d(V_j)_{\mathbb{H}}$$

$$+\frac{((1+\epsilon)-\lambda)(2+\epsilon)}{2}\int_{\mathbb{H}^{3+\epsilon}}\sum\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho^{\lambda+2}(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$\geq \frac{((1+\epsilon)-\lambda)^2}{4}\int_{\mathbb{H}^{3+\epsilon}}\sum \frac{|f_j|^2}{\rho^{\lambda+2}(x)}d(V_j)_{\mathbb{H}}.$$

Proof. $\left(r^{1+\epsilon-\lambda}, r^{1+\epsilon-\lambda}\frac{((1+\epsilon)-\lambda)^2}{4}\frac{1}{r^2}\right)$ is a Bessel pair on $(0,\infty)$ with $\varphi(r) = r^{\frac{-(1+\epsilon)+\lambda}{2}}$.

Corollary 3.3 [43]. We have

$$\int_{0<\rho(x)<1+\epsilon}\sum \frac{|\nabla_{\mathbb{H}} f_j|^2}{\rho^{1+\epsilon}(x)}d(V_j)_{\mathbb{H}} - \frac{z_0^2}{(1+\epsilon)^2}\int_{0<\rho(x)<1+\epsilon}\sum \frac{|f_j|^2}{\rho^{1+\epsilon}(x)}d(V_j)_{\mathbb{H}}$$

$$= \int_{0<\rho(x)<1+\epsilon}\sum \frac{J_0^2\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}{\rho^{1+\epsilon}(x)}\left|\nabla_{\mathbb{H}}\left(\frac{f_j}{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}\right)\right|^2 d(V_j)_{\mathbb{H}}$$

$$-(2+\epsilon)\frac{z_0}{1+\epsilon}\int_{0<\rho(x)<1+\epsilon}\sum \frac{J_0'\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho^{2+\epsilon}(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

and

$$\int_{0<\rho(x)<1+\epsilon}\sum \frac{|\partial_\rho f_j|^2}{\rho^{2+\epsilon}(x)}d(V_j)_{\mathbb{H}} - \frac{z_0^2}{(1+\epsilon)^2}\int_{0<\rho(x)<1+\epsilon}\sum \frac{|f_j|^2}{\rho^{1+\epsilon}(x)}d(V_j)_{\mathbb{H}}$$

$$= \int_{0<\rho(x)<1+\epsilon}\sum \frac{J_0^2\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}{\rho^{1+\epsilon}(x)}\left|\partial_\rho\left(\frac{f_j}{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}\right)\right|^2 d(V_j)_{\mathbb{H}}$$

$$-(2+\epsilon)\frac{z_0}{1+\epsilon}\int_{0<\rho(x)<1+\epsilon}\sum \frac{J_0'\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho^{2+\epsilon}(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}.$$

As a consequence of these identities, we get that

$$\int_{0<\rho(x)<1+\epsilon}\sum \frac{|\nabla_{\mathbb{H}} f_j|^2}{\rho^{1+\epsilon}(x)}d(V_j)_{\mathbb{H}} \geq \int_{0<\rho(x)<1+\epsilon}\sum \frac{|\partial_\rho f_j|^2}{\rho^{1+\epsilon}(x)}d(V_j)_{\mathbb{H}}$$

$$\geq \frac{z_0^2}{(1+\epsilon)^2}\int_{0<\rho(x)<1+\epsilon}\sum \frac{|f_j|^2}{\rho^{1+\epsilon}(x)}d(V_j)_{\mathbb{H}}$$

$$-(2+\epsilon)\frac{z_0}{1+\epsilon}\int_{0<\rho(x)<1+\epsilon}\sum \frac{J_0'\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}{J_0\left(\frac{z_0}{1+\epsilon}\rho(x)\right)}\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho^{2+\epsilon}(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$\geq \frac{z_0^2}{(1+\epsilon)^2}\int_{0<\rho(x)<1+\epsilon}\sum \frac{|f_j|^2}{\rho^{1+\epsilon}(x)}d(V_j)_{\mathbb{H}}.$$

Proof. $\left(r, r\frac{z_0^2}{(1+\epsilon)^2}\right)$ is a Bessel pair on $(0,1+\epsilon)$ with $\varphi(r) = J_0\left(\frac{z_0}{1+\epsilon}r\right)$ and $\varphi'(r) = \frac{z_0}{1+\epsilon}J_0'\left(\frac{z_0}{1+\epsilon}r\right)$

Proof of Theorem 1.5 [43]. We note that $\left(r^{2+\epsilon-\lambda}, r^{2+\epsilon-\lambda}\left[\left(\frac{((1+\epsilon)-\lambda)^2}{4} - \alpha^2\right)\frac{1}{r^2} + \frac{z_\alpha^2}{(1+\epsilon)^2}\right]\right)$ on $(0,1+\epsilon)$ with $\varphi(r) = r^{\frac{\lambda-(1+\epsilon)}{2}}J_\alpha\left(\frac{z_\alpha}{1+\epsilon}r\right), 0 \le \alpha \le \frac{1+\epsilon-\lambda}{2}$. Here z_α is the first zero of the Bessel function $J_\alpha(z)$. Now, we can apply Theorem 3.2 to obtain the desired results.

Corollary 3.4 [43]. We have

$$\int_{0<\rho(x)<1+\epsilon} \sum \frac{\left|\nabla_{\mathbb{H}} f_j\right|^2}{\rho^{1+\epsilon}(x)} d(V_j)_{\mathbb{H}} - \frac{1}{4}\int_{0<\rho(x)<1+\epsilon} \sum \frac{|f_j|^2}{\rho^{3+\epsilon}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2} d(V_j)_{\mathbb{H}}$$

$$= \int_{0<\rho(x)<1+\epsilon} \sum \frac{1}{\rho^{1+\epsilon}(x)}\left|\ln\frac{\rho(x)}{1+\epsilon}\right|\left|\nabla_{\mathbb{H}}\left(\frac{f_j}{\sqrt{\left|\ln\frac{|x|}{1+\epsilon}\right|}}\right)\right|^2 d(V_j)_{\mathbb{H}}$$

$$+ \int_{0<\rho(x)<1+\epsilon} \sum \frac{1}{\rho^{1+\epsilon}(x)} \frac{1}{2\left|\ln\frac{\rho(x)}{1+\epsilon}\right|} \frac{\rho(x)\cosh\rho(x) - \sinh\rho(x)}{\rho^2(x)\sinh\rho(x)} |f_j|^2 d(V_j)_{\mathbb{H}}$$

and

$$\int_{0<\rho(x)<1+\epsilon} \sum \frac{\left|\partial_\rho f_j\right|^2}{\rho^{1+\epsilon}(x)} d(V_j)_{\mathbb{H}} - \frac{1}{4}\int_{0<\rho(x)<1+\epsilon} \sum \frac{|f_j|^2}{\rho^{3+\epsilon}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2} d(V_j)_{\mathbb{H}}$$

$$= \int_{0<\rho(x)<1+\epsilon} \sum \frac{1}{\rho^{1+\epsilon}(x)}\left|\ln\frac{\rho(x)}{1+\epsilon}\right|\left|\partial_\rho\left(\frac{f_j}{\sqrt{\left|\ln\frac{|x|}{1+\epsilon}\right|}}\right)\right|^2 d(V_j)_{\mathbb{H}}$$

$$+ \int_{0<\rho(x)<1+\epsilon} \sum \frac{1}{\rho^{1+\epsilon}(x)} \frac{1}{2\left|\ln\frac{\rho(x)}{1+\epsilon}\right|} \frac{\rho(x)\cosh\rho(x) - \sinh\rho(x)}{\rho^2(x)\sinh\rho(x)} |f_j|^2 d(V_j)_{\mathbb{H}}$$

As a consequence of these identities, we get that

$$\int_{0<\rho(x)<1+\epsilon} \sum \frac{\left|\nabla_{\mathbb{H}} f_j\right|^2}{\rho^{1+\epsilon}(x)} d(V_j)_{\mathbb{H}} \ge \int_{0<\rho(x)<1+\epsilon} \sum \frac{\left|\partial_\rho f_j\right|^2}{\rho^{1+\epsilon}(x)} d(V_j)_{\mathbb{H}}$$

$$\geq \frac{1}{4}\int_{0<\rho(x)<1+\epsilon}\sum \frac{|f_j|^2}{\rho^{3+\epsilon}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2} d(V_j)_{\mathbb{H}}$$

$$+\int_{0<\rho(x)<1+\epsilon}\sum \frac{1}{\rho^{1+\epsilon}(x)}\frac{1}{2\left|\ln\frac{\rho(x)}{1+\epsilon}\right|}\frac{\rho(x)\cosh\rho(x)-\sinh\rho(x)}{\rho^2(x)\sinh\rho(x)}|f_j|^2 d(V_j)_{\mathbb{H}}$$

$$\geq \frac{1}{4}\int_{0<\rho(x)<1+\epsilon}\sum \frac{|f_j|^2}{\rho^{3+\epsilon}(x)\left|\ln\frac{\rho(x)}{1+\epsilon}\right|^2} d(V_j)_{\mathbb{H}}.$$

Proof. $\left(r, \frac{1}{4r\left|\ln\frac{r}{1+\epsilon}\right|^2}\right)$ is a Bessel pair on $(0,1+\epsilon)$ with $\varphi(r)=\sqrt{\left|\ln\frac{r}{1+\epsilon}\right|}$ and $\varphi'(r)=-\frac{1}{2r\sqrt{\left|\ln\frac{r}{1+\epsilon}\right|}}$

4. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 [43]. Let $f_j(x)=\varphi(\rho(x))v(x)$, then

$$\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))\left|\nabla_g f_j\right|_g^2 d(V_j)_g$$

$$=\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))\left|\nabla_g(\varphi(\rho(x))v(x))\right|_g^2 d(V_j)_g$$

$$=\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))|\varphi^2(\rho(x))|\left|\nabla_g v\right|_g^2 d(V_j)_g$$

$$+\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))|v^2(x)|\left|\nabla_g\varphi(\rho(x))\right|_g^2 d(V_j)_g$$

$$+\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))\varphi(\rho(x))\left\langle\nabla_g v^2,\nabla_g\varphi(\rho(x))\right\rangle_g d(V_j)_g$$

$$=\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))|\varphi^2(\rho(x))|\left|\nabla_g v\right|_g^2 dV_g$$

$$+\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))|v^2(x)||\varphi'(\rho(x))|^2 d(V_j)_g$$

$$+\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))\left\langle\nabla_g v^2,\nabla_g\rho(x)\right\rangle_g d(V_j)_g$$

Now, using the divergence theorem, we get

$$\int_{B_{1+\epsilon}(O)}\sum V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))\left\langle\nabla_g v^2,\nabla_g\rho(x)\right\rangle_g dV_g$$

$$= -\int_{B_{1+\epsilon}(O)} \sum v^2(x)\mathrm{div}\big(V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))\nabla_g\rho(x)\big)$$

$$= -\int_{B_{1+\epsilon}(O)} \sum v^2(x)V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))\Delta_g\rho(x)d(V_j)_g$$

$$-\int_{B_{1+\epsilon}(O)} \sum v^2(x)V_j'(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))d(V_j)_g$$

$$-\int_{B_{1+\epsilon}(O)} \sum v^2(x)V_j(\rho(x))\varphi'(\rho(x))\varphi'(\rho(x))d(V_j)_g$$

$$-\int_{B_{1+\epsilon}(O)} \sum v^2(x)V_j(\rho(x))\varphi(\rho(x))\varphi''(\rho(x))d(V_j)_g$$

Noting that (see [22,4.B.2])

$$\Delta_g\rho(x) = \frac{(3+\epsilon)-1}{\rho(x)} + \frac{J'(u,\rho(x))}{J(u,\rho(x))}.$$

Hence

$$\int_{B_{1+\epsilon}(O)} V_j \sum (\rho(x))|\nabla_g f_j|_g^2 d(V_j)_g - \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|\varphi^2(\rho(x))||\nabla_g v|_g^2 d(V_j)_g$$

$$= -\int_{B_{1+\epsilon}(O)} \sum \varphi(\rho(x))v^2(x)\left[\begin{matrix} V_j(\rho(x))\varphi'(\rho(x))\dfrac{2+\epsilon}{\rho(x)} \\ +V_j'(\rho(x))\varphi'(\rho(x)) + V_j(\rho(x))\varphi''(\rho(x)) \end{matrix}\right] d(V_j)_g$$

$$-\int_{B_{1+\epsilon}(O)} \sum v^2(x)V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))\frac{J'(u,\rho(x))}{J(u,\rho(x))}d(V_j)_g$$

$$= \int_{B_{1+\epsilon}(O)} \sum \varphi^2\big(\rho(x)\big)v^2(x)W_j(\rho(x))$$

$$-\int_{B_{1+\epsilon}(O)} \sum v^2(x)V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))\frac{J'(u,\rho(x))}{J(u,\rho(x))}d(V_j)_g$$

$$= \int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f|^2 d(V_j)_g$$

$$-\int_{B_{1+\epsilon}(O)} \sum v^2(x)V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x))\frac{J'(u,\rho(x))}{J(u,\rho(x))}d(V_j)_g$$

$$= \int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f_j|^2 d(V_j)_g$$

$$-\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))|f_j|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))}\frac{J'(u,\rho(x))}{J(u,\rho(x))}d(V_j)_g.$$

Now, denote $F_j(y) = f_j(\exp_O(y)), \Phi(y) = \varphi(\exp_O(y))$ and $\Psi(y) = v(\exp_O(y))$. Then using the polar coordinate we get

$$\int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f_j|^2 d(V_j)_g$$
$$= \int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} W_j(\rho)\Phi(\rho)\Phi(\rho)\Psi^2(\rho u)J(u,\rho)d\rho du$$
$$= \int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \partial_\rho\left(\rho^{2+\epsilon} V_j(\rho)\,\partial_\rho\Phi(\rho)\right)\Phi(\rho)\Psi^2(\rho u)J(u,\rho)d\rho du$$
$$= \int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} V_j(\rho)\,\partial_\rho\Phi(\rho)\,\partial_\rho[\Phi(\rho)\Psi^2(\rho u)J(u,\rho)]d\rho du$$
$$= \int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} V_j(\rho)\left(\partial_\rho\Phi(\rho)\right)^2\Psi^2(\rho u)J(u,\rho)d\rho du$$
$$+2\int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} V_j(\rho)\,\partial_\rho\Phi(\rho)\Phi(\rho)\,\partial_\rho\Psi(\rho u)\Psi(\rho u)J(u,\rho)d\rho du$$
$$+\int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} V_j(\rho)\,\partial_\rho\Phi(\rho)\Phi(\rho)\Psi^2(\rho u)\,\partial_\rho J(u,\rho)d\rho du.$$

Hence, we have

$$\int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x))|f_j|^2 d(V_j)_g$$
$$= \int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} V_j(\rho)\left|\Psi(\rho u)\,\partial_\rho\Phi(\rho) + \Phi(\rho)\,\partial_\rho\Psi(\rho u)\right|^2 J(u,\rho)d\rho du$$
$$- \int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} V_j(\rho)\left|\Phi(\rho)\,\partial_\rho\Psi(\rho u)\right|^2 J(u,\rho)d\rho du$$
$$+ \int_{\mathbb{S}^{2+\epsilon}} \int_0^{1+\epsilon} \sum \rho^{2+\epsilon} V(\rho)\,\partial_\rho\Phi(\rho)\Phi(\rho)\Psi^2(\rho u)\,\partial_\rho J(u,\rho)d\rho du$$
$$= \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\left|\partial_\rho f_j\right|^2 d(V_j)_g - \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x))\left|\partial_\rho\left(\frac{f_j}{\varphi(\rho(x))}\right)\right|^2 \varphi^2(\rho(x))d(V_j$$
$$+ \int_{B_{1+\epsilon}(O)} \sum \left(\frac{f_j}{\varphi(\rho(x))}\right)^2 \varphi(\rho(x))\varphi'(\rho(x))\frac{J'(u,\rho(x))}{J(u,\rho(x))}d(V_j)_g$$

Proof of Theorem 1.2 [43]. Let $f_j(x) - f_j(\exp_O((1+\epsilon)u)) = \varphi(\rho(x))v(x)$. Then proceed as in the proof of Theorem 1.1, we get

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \left|\nabla_g \left(f_j - f_j(\exp_O((1+\epsilon)u))\right)\right|_g^2 d\left(V_j\right)_g$$

$$= \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \left|\nabla_g (\varphi(\rho(x)) v(x))\right|_g^2 d\left(V_j\right)_g$$

$$= \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) |\varphi^2(\rho(x))| \left|\nabla_g v\right|_g^2 dV_g + \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) |v^2| \left|\nabla_g \varphi(\rho(x))\right|_g^2 d\left(V_j\right)_g$$

$$+ \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \varphi(\rho(x)) \left\langle \nabla_g v^2, \nabla_g \varphi(\rho(x))\right\rangle_g dV_g$$

$$= \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) |\varphi^2(\rho(x))| \left|\nabla_g v\right|_g^2 d\left(V_j\right)_g + \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) |v^2| |\varphi'(\rho(x))|^2 d\left(V_j\right)_g$$

$$+ \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \left\langle \nabla_g v^2, \nabla_g \rho(x)\right\rangle_g d\left(V_j\right)_g .$$

Now, using the divergence theorem, we get

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \left\langle \nabla_g v^2, \nabla_g \rho(x)\right\rangle_g d\left(V_j\right)_g$$

$$= -\int_{B_{1+\epsilon}(O)} \sum v^2 \operatorname{div}\left(V(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \nabla_g \rho(x)\right) d\left(V_j\right)_g$$

$$- \int_{\partial B_{1+\epsilon}(O)} \sum v^2 V_j\left(\rho(x)\right) \varphi\left(\rho(x)\right) \varphi'\left(\rho(x)\right) \frac{\partial \rho}{\partial \nu}(x) dS_g .$$

By the assumption (C) on f_j, we get

$$\int_{\partial B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \frac{\partial \rho}{\partial \nu}(x) dS_g = 0.$$

Hence

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \left\langle \nabla_g v^2, \nabla_g \rho(x)\right\rangle_g d\left(V_j\right)_g$$

$$= -\int_{B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \Delta_g \rho(x) d\left(V_j\right)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum v^2 V_j'(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) d\left(V_j\right)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x)) \varphi'(\rho(x)) \varphi'(\rho(x)) d\left(V_j\right)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x)) \varphi(\rho(x)) \varphi''(\rho(x)) d\left(V_j\right)_g$$

Again, using

$$\Delta_g \rho(x) = \frac{2+\epsilon}{\rho(x)} + \frac{J'(u,\rho)}{J(u,\rho)}$$

we get

$$\int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \left|\nabla_g \left(f_j - f_j(\exp_O((1+\epsilon)u))\right)\right|_g^2 d\left(V_j\right)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum V_j(\rho(x)) |\varphi^2(\rho(x))| \left|\nabla_g v\right|_g^2 d\left(V_j\right)_g$$

$$= - \int_{B_{1+\epsilon}(O)} \sum \varphi(\rho(x)) v^2 \begin{bmatrix} V_j(\rho(x))\varphi'(\rho(x)) \dfrac{2+\epsilon}{\rho(x)} \\ +V_j'(\rho(x))\varphi'(\rho(x)) + V_j(\rho(x))\varphi''(\rho(x)) \end{bmatrix} d\left(V_j\right)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x)) \frac{J'(u,\rho)}{J(u,\rho)} d\left(V_j\right)_g$$

$$= \int_{B_{1+\epsilon}(O)} \sum \varphi^2\big(\rho(x)\big) v^2 W_j(\rho(x)) - \int_{B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x)) \frac{J'(u,\rho)}{J(u,\rho)} d(V$$

$$= \int_{B_{1+\epsilon}(O)} \sum W_j(\rho(x)) \left|f_j - f_j(\exp_O((1+\epsilon)u))\right|^2 d\left(V_j\right)_g$$

$$- \int_{B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x)) \frac{J'(u,\rho)}{J(u,\rho)} d\left(V_j\right)_g .$$

Similarly,

$$\int_{\mathbb{M}\setminus B_{1+\epsilon}(O)} \sum V_j(\rho(x)) \left|\nabla_g \left(f_j - f_j(\exp_O((1+\epsilon)u))\right)\right|_g^2 d\left(V_j\right)_g$$

$$- \int_{\mathbb{M}\setminus B_{1+\epsilon}(O)} \sum V_j(\rho(x)) |\varphi^2(\rho(x))| \left|\nabla_g \left(\frac{f_j - f_j(\exp_O((1+\epsilon)u))}{\varphi(\rho(x))}\right)\right|_g^2 d\left(V_j\right)_g$$

$$= \int_{\mathbb{M}\setminus B_{1+\epsilon}(O)} \sum W_j(\rho(x)) \left|f_j - f_j(\exp_O((1+\epsilon)u))\right|^2 d\left(V_j\right)_g$$

$$- \int_{\mathbb{M}\setminus B_{1+\epsilon}(O)} \sum v^2 V_j(\rho(x))\varphi(\rho(x))\varphi'(\rho(x)) \frac{J'(u,\rho)}{J(u,\rho)} d\left(V_j\right)_g .$$

Therefore

$$\int_{\mathbb{M}} \sum V_j(\rho(x)) \left|\nabla_g \left(f_j - f_j(\exp_O((1+\epsilon)u))\right)\right|_g^2 dx$$

$$- \int_{\mathbb{M}} \sum W_j(\rho(x)) \left| f_j - f_j(\exp_O((1+\epsilon)u)) \right|^2 d(V_j)_g$$

$$= \int_{\mathbb{M}} \sum V_j(\rho(x)) \varphi^2(\rho(x)) \left| \nabla_g \left(\frac{f_j - f_j(\exp_O((1+\epsilon)u))}{\varphi(\rho(x))} \right) \right|_g^2 dx$$

$$- \int_{\mathbb{M}} \sum V_j(\rho(x)) \left| f_j - f_j(\exp_O((1+\epsilon)u)) \right|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} \frac{J'(u,\rho)}{J(u,\rho)} d(V_j)_g$$

Similarly,

$$\int_{\mathbb{M}} \sum V_j(\rho(x)) \left| \partial_\rho \left(f_j - f_j(\exp_O((1+\epsilon)u)) \right) \right|_g^2 dx$$

$$- \int_{\mathbb{M}} \sum W_j(\rho(x)) \left| f_j - f_j(\exp_O((1+\epsilon)u)) \right|^2 d(V_j)_g$$

$$= \int_{\mathbb{M}} \sum V_j(\rho(x)) \varphi^2(\rho(x)) \left| \partial_\rho \left(\frac{f_j - f_j(\exp_O((1+\epsilon)u))}{\varphi(\rho(x))} \right) \right|^2 dx$$

$$- \int_{\mathbb{M}} \sum V_j(\rho(x)) \left| f_j - f_j(\exp_O((1+\epsilon)u)) \right|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} \frac{J'(u,\rho)}{J(u,\rho)} d(V_j)_g$$

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